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QUASI-STATIONARY DISTRIBUTIONS AND DIFFUSION MODELS IN POPULATION DYNAMICS

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ABSTRACT. In this paper, we study quasi-stationarity for a large class of Kolmogorov diffusions. The main novelty here is that we allow the drift to go to $-\infty$ at the origin, and the diffusion to have an entrance boundary at $+\infty$. These diffusions arise as images, by a deterministic map, of generalized Feller diffusions, which themselves are obtained as limits of rescaled birth–death processes. Generalized Feller diffusions take nonnegative values and are absorbed at zero in finite time with probability 1. An important example is the logistic Feller diffusion.

We give sufficient conditions on the drift near 0 and near $+\infty$ for the existence of quasi-stationary distributions, as well as rate of convergence in the Yaglom limit and existence of the Q-process. We also show that under these conditions, there is exactly one quasi-stationary distribution, and that this distribution attracts all initial distributions under the conditional evolution, if and only if $+\infty$ is an entrance boundary. In particular this gives a sufficient condition for the uniqueness of quasi-stationary distributions. In the proofs spectral theory plays an important role on L^2 of the reference measure for the killed process.

Key words. quasi-stationary distribution, birth–death process, population dynamics, logistic growth, generalized Feller diffusion, Yaglom limit, convergence rate, Q-process, entrance boundary at infinity.

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1. **Introduction**

The main motivation of this work is the existence, uniqueness and domain of attraction of quasi-stationary distributions for some diffusion models arising from population dynamics. After a change of variable, the problem is stated in the framework of Kolmogorov diffusion processes with a drift behaving like $-1/2x$ near the origin. Hence, we shall study quasi-stationarity for the larger class of one-dimensional Kolmogorov diffusions (drifted Brownian motions), with drift possibly exploding at the origin.

Consider a one-dimensional drifted Brownian motion on $(0, \infty)$

$$dX_t = dB_t - q(X_t)\,dt \quad , \quad X_0 = x > 0 \tag{1.1}$$

where q is defined and C^1 on $(0, \infty)$ and $(B_t; t \geq 0)$ is a standard one-dimensional Brownian motion. In particular q is allowed to explode at the origin. A pathwise unique solution of

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(1.1) exists up to the explosion time τ. We denote T_y the first time the process hits $y \in (0,\infty)$ (see [14] chapter VI section 3) before the explosion

$$T_y = \inf\{0 \le t < \tau : X_t = y\}$$

We denote by $T_\infty = \lim_{n\to\infty} T_n$ and $T_0 = \lim_{n\to\infty} T_{1/n}$. Since q is regular in $(0,\infty)$ then $\tau = T_0 \wedge T_\infty$. The law of the process starting from X_0 with distribution ν will be denoted by $\mathbb{P}_\nu$. A *quasi-stationary distribution* (in short *q.s.d.*) for X is a probability measure ν supported on $(0,\infty)$ satisfying for all $t \ge 0$

$$\mathbb{P}_\nu(X_t \in A \mid T_0 > t) = \nu(A), \ \ \forall \text{ Borel set } A \subseteq (0,\infty). \tag{1.2}$$

By definition a *q.s.d.* is a fixed point of the conditional evolution. The *Yaglom limit* π is defined as the limit in distribution

$$\pi(\bullet) = \lim_{t\to\infty} \mathbb{P}_x(X_t \in \bullet \mid T_0 > t),$$

provided this limit exists and is independent of the initial condition x. The Yaglom limit is a *q.s.d.* (see Lemma 7.2).

We will also study the existence of the so-called Q-process which is obtained as the law of the process X conditioned to be never extinct, and it is defined as follows. For any $s \ge 0$ and any Borel set $B \subseteq C([0,s])$ consider

$$\mathbb{Q}_x(X \in B) = \lim_{t\to\infty} \mathbb{P}_x(X \in B \mid T_0 > t).$$

When it exists, this limit procedure defines the law of a diffusion that never reaches 0.

The reason for studying such diffusion processes with a possibly exploding drift at the origin comes from our interest in the following *generalized Feller diffusion processes*

$$dZ_t = \sqrt{\gamma Z_t}dB_t + h(Z_t)dt, \ Z_0 = z > 0, \tag{1.3}$$

where h is a nice function satisfying $h(0) = 0$.

Notice that $z = 0$ is an absorbing state for Z. This means that if $Z_0 = 0$ then $Z_t = 0$ for all t is the unique solution of (1.3) (see [14]).

If we define $X_t = 2\sqrt{Z_t/\gamma}$ then

$$dX_t = dB_t - \frac{1}{X_t}\left(\frac{1}{2} - \frac{2}{\gamma}\, h\left(\frac{\gamma X_t^2}{4}\right)\right) dt \ \ , \ X_0 = x = 2\sqrt{z/\gamma} > 0\,, \tag{1.4}$$

so that X is a drifted Brownian motion as in (1.1) where $q(x)$ behaves like $1/2x$ near the origin. The process Z is obtained after rescaling some sequences of birth–death processes arising from population dynamics.

Of particular interest is the case $h(z) = rz - cz^2$ (logistic case), for which we obtain

$$q(x) = \frac{1}{2x} - \frac{rx}{2} + \frac{c\gamma x^3}{8}\,.$$

A complete description of these models is performed in the final section of the paper, where their biological meaning is also discussed. Of course quasi-stationary distributions for Z and X are related by an immediate change of variables, so that the results on X can be immediately translated to results on Z.

The study of quasi-stationarity is a long standing problem (see [29] for a regularly updated extensive bibliography and [10, 13, 33] for the Markov chain case). For Kolmogorov diffusions, the theory started with Mandl's paper [26] in 1961, and was then developed by many authors (see in particular [5, 27, 35]). All these works assume Mandl's conditions, which are not satisfied in the situation described above, since in particular the drift is not bounded near 0. It is worth noticing that the behavior of q at infinity also may violate Mandl's conditions, since in the logistic case for instance,

$$\int_1^\infty e^{-Q(z)} \int_1^z e^{Q(y)} dy\, dz < \infty,$$

where $Q(y) := 2 \int_1^y q(x)\, dx$.

This unusual situation prevents us from using earlier results on *q.s.d.*'s of solutions of Kolmogorov equations. Hence we are led to develop new techniques to cope with this situation.

In Section 2 we start with the study of a general Kolmogorov diffusion process on the half line and introduce the hypothesis (H1) that ensures to reach 0 in finite time with probability 1. Then we introduce the measure μ, not necessarily finite, defined as

$$\mu(dy) := e^{-Q(y)}\, dy,$$

which is the speed measure of X. We describe the Girsanov transform and show how to use it in order to obtain $\mathbb{L}^2(\mu)$ estimates for the heat kernel (Theorem 2.3). The key is the following: starting from any $x > 0$, the law of the process at time t is absolutely continuous with respect to μ with a density belonging to $\mathbb{L}^2(\mu)$ (and explicit bounds). In the present paper we work in $\mathbb{L}^p(\mu)$ spaces rather than $\mathbb{L}^p(dx)$, since it greatly simplifies the presentation of the spectral theory.

This spectral theory is done in Section 3, where we introduce the hypothesis (H2):

$$\lim_{x\to\infty} q^2(x) - q'(x) = \infty \ , \ C := - \inf_{x\in(0,\infty)} q^2(x) - q'(x) < \infty.$$

This hypothesis ensures the discreteness of the spectrum (Theorem 3.2). The ground state η_1 (eigenfunction associated to the bottom of the spectrum) can be chosen nonnegative, even positive as we will see, and furnishes the natural candidate $\eta_1 d\mu$ for a *q.s.d.*. The only thing to check is that $\eta_1 \in \mathbb{L}^1(\mu)$ which is not immediate since μ is possibly unbounded.

Section 4 gives some sharper properties of the eigenfunctions defined in the previous section, using in particular properties of the Dirichlet heat kernel. More specifically, we introduce two independent hypotheses, (H3) and (H4), either of which ensures that the eigenfunctions belong to $\mathbb{L}^1(\mu)$ (Propositions 4.3 and 4.4). Hypothesis (H3) is

$$\int_0^1 \frac{1}{q^2(y) - q'(y) + C + 2}\, \mu(dy) < \infty,$$

and (H4) is

$$\int_1^\infty e^{-Q(y)}\, dy < \infty \ , \ \int_0^1 y e^{-Q(y)/2}\, dy < \infty.$$

Section 5 contains the proofs of the existence of the Yaglom limit (Theorem 5.2) as well as the exponential decay to equilibrium (Proposition 5.5), under hypotheses (H1) and (H2), together with either (H3) or (H4). Section 6 contains the results on the Q-process (Corollaries 6.1 and 6.2).

In Section 7 we introduce condition (H5) which is equivalent to the existence of an entrance law at $+\infty$, that is, the repelling force at infinity imposes to the process starting from infinity to reach any finite interval in finite time. The process is then said to "come down from infinity". We show that the process comes down from infinity if and only if there exists a unique *q.s.d.* which attracts any initial law under the conditional evolution (Theorem 7.3). In particular this theorem gives sufficient conditions for uniqueness of *q.s.d.*'s. In the context of birth and death chains the equivalence between uniqueness of a *q.s.d.* and "come down from infinity" has been proved in [8, Theorem 3.2].

The final section contains the description of the underlying biological models, as well as the application of the whole theory developed in the previous sections to these models (Theorem 8.2).

In the following statement (which is basically Theorem 8.2), we record the main results of this paper in terms of the generalized Feller diffusion solution of (1.3). We say that h satisfies the condition (HH) if

$$(i)\ \lim_{x\to\infty} \frac{h(x)}{\sqrt{x}} = -\infty, \qquad (ii)\ \lim_{x\to\infty} \frac{xh'(x)}{h(x)^2} = 0.$$

Theorem 1.1. *If h satisfies (HH), then for all initial laws with bounded support, the law of Z_t conditioned on $\{Z_t \neq 0\}$ converges exponentially fast to a probability measure ν, called the Yaglom limit.*

The process Z conditioned to be never extinct is well-defined and is called the Q-process. The Q-process converges in distribution, to its unique invariant probability measure. This probability measure is absolutely continuous w.r.t. ν with a nondecreasing Radon–Nikodym derivative.

If in addition, the following integrability condition is satisfied

$$\int_1^\infty \frac{dx}{-h(x)} < \infty,$$

then Z comes down from infinity and the convergence of the conditional one-dimensional distributions holds for all initial laws, so that the Yaglom limit ν is the unique quasi-stationary distribution.

2. One dimensional diffusion processes on the positive half line

Associated to q we consider the functions

$$\Lambda(x) = \int_1^x e^{Q(y)}\,dy \qquad \text{and} \qquad \kappa(x) = \int_1^x e^{Q(y)} \left(\int_1^y e^{-Q(z)}\,dz \right) dy\,, \tag{2.1}$$

where we recall that $Q(y) = \int_1^y 2q(u)du$. Notice that Λ is the scale function for X.

For most of the results in this paper we shall assume sure absorption at zero, that is

Hypothesis (H1): for all $x > 0$, $\quad \mathbb{P}_x(\tau = T_0 < T_\infty) = 1\,.$ (2.2)

It is well known (see e.g. [14] chapter VI Theorem 3.2) that (2.2) holds if and only if

$$\Lambda(\infty) = \infty \ \text{ and } \ \kappa(0^+) < \infty\,. \tag{2.3}$$

We notice that (H1) can be written as $\mathbb{P}_x(\lim_{t\to\infty} X_{t\wedge\tau} = 0) = 1$.

Example 2.1. The main cases that we are interested in are the following ones.

(1) When X is defined by (1.4) associated to the generalized Feller diffusion Z. It is direct to show that $Q(x)$ behaves like $\log(x)$ near 0 hence $\kappa(0^+) < \infty$. The logistic Feller diffusion corresponds to $h(z) = rz - cz^2$ for some constants c and r. It is easily seen that (2.3) is satisfied in this case provided $c > 0$ or $c = 0$ and $r < 0$.
(2) When the drift is bounded near 0, in which case $\kappa(0^+) < \infty$.

$\diamondsuit$

We shall now discuss some properties of the law of X up to T_0. The first result is a Girsanov type result.

Proposition 2.2. *Assume (H1). For any Borel bounded function $F : C([0,t],(0,\infty)) \to \mathbb{R}$ it holds*

$$\mathbb{E}_x\left[F(X)\,\mathbb{1}_{t<T_0}\right] \;=\; \mathbb{E}^{\mathbb{W}_x}\left[F(\omega)\,\mathbb{1}_{t<T_0(\omega)}\,\exp\left(\frac{1}{2}\,Q(x) - \frac{1}{2}\,Q(\omega_t) - \frac{1}{2}\int_0^t (q^2 - q')(\omega_s)ds\right)\right]$$

where $\mathbb{E}^{\mathbb{W}_x}$ denotes the expectation w.r.t. the Wiener measure starting from x, and $\mathbb{E}_x$ denotes the expectation with respect to the law of X starting also from x.

Proof. It is enough to show the result for F nonnegative and bounded. Let $x > 0$ and consider $\varepsilon \in (0,1)$ such that $\varepsilon \le x \le 1/\varepsilon$. Also we define $\tau_\varepsilon = T_\varepsilon \wedge T_{1/\varepsilon}$. Choose some ψ_ε which is a nonnegative C^∞ function with compact support, included in $]\varepsilon/2, 2/\varepsilon[$ such that $\psi_\varepsilon(u) = 1$ if $\varepsilon \le u \le 1/\varepsilon$. The law of the diffusion (1.1) coincides up to τ_ε with the law of a similar diffusion process X^ε obtained by replacing q with the cutoff $q_\varepsilon = q\psi_\varepsilon$. For the latter we may apply the Novikov criterion ensuring that the law of X^ε is given via the Girsanov formula. Hence

$$\begin{aligned}
\mathbb{E}_x\left[F(X)\,\mathbb{1}_{t<\tau_\varepsilon}\right] &= \mathbb{E}^{W_x}\left[F(\omega)\,\mathbb{1}_{t<\tau_\varepsilon(\omega)}\,\exp\left(\int_0^t -q_\varepsilon(\omega_s)d\omega_s - \frac{1}{2}\int_0^t (q_\varepsilon)^2(\omega_s)ds\right)\right]\\
&= \mathbb{E}^{W_x}\left[F(\omega)\,\mathbb{1}_{t<\tau_\varepsilon(\omega)}\,\exp\left(\int_0^t -q(\omega_s)d\omega_s - \frac{1}{2}\int_0^t q^2(\omega_s)ds\right)\right]\\
&= \mathbb{E}^{W_x}\left[F(\omega)\,\mathbb{1}_{t<\tau_\varepsilon(\omega)}\,\exp\left(\frac{1}{2}\,Q(x) - \frac{1}{2}\,Q(\omega_t) - \frac{1}{2}\int_0^t (q^2 - q')(\omega_s)ds\right)\right].
\end{aligned}$$

The last equality is obtained integrating by parts the stochastic integral. But $\mathbb{1}_{t<\tau_\varepsilon}$ is non-decreasing in ε and converges almost surely to $\mathbb{1}_{t<T_0}$ both for $\mathbb{P}_x$ (thanks to (H1)) and $\mathbb{W}_x$. It remains to use Lebesgue monotone convergence theorem to finish the proof. $\square$

The next theorem is inspired by the calculation in Theorem 3.2.7 of [32]. It will be useful to introduce the following measure defined on $(0,\infty)$

$$\mu(dy) := e^{-Q(y)}\,dy\,. \tag{2.4}$$

Note that μ is not necessarily finite.

Theorem 2.3. *Assume (H1). For all $x > 0$ and all $t > 0$ there exists some density $r(t,x,.)$ that satisfies*

$$\mathbb{E}_x[f(X_t)\,\mathbb{1}_{t<T_0}] = \int_0^\infty f(y)\,r(t,x,y)\,\mu(dy)$$

for all bounded Borel f.

If in addition there exists some $C > 0$ such that $q^2(y) - q'(y) \geq -C$ for all $y > 0$, then for all $t > 0$ and all $x > 0$,

$$\int_0^\infty r^2(t,x,y)\,\mu(dy) \;\leq (1/2\pi t)^{\frac{1}{2}}\, e^{Ct}\, e^{Q(x)}\,.$$

Proof. Define

$$G(\omega) = \mathbb{I}_{t<T_0(\omega)} \, \exp\left(\frac{1}{2}\,Q(\omega_0) - \frac{1}{2}\,Q(\omega_t) - \frac{1}{2}\int_0^t (q^2 - q')(\omega_s) ds\right)\,.$$

Denote by

$$e^{-v(t,x,y)} = (2\pi t)^{-\frac{1}{2}}\, \exp\left(-\frac{(x-y)^2}{2t}\right)$$

the density at time t of the Brownian motion starting from x. According to Proposition 2.2 we have

$$\begin{aligned}
\mathbb{E}_x[f(X_t)\,\mathbb{I}_{t<T_0}] &= \mathbb{E}^{\mathbb{W}_x}[f(\omega_t)\mathbb{E}^{\mathbb{W}_x}[G|\omega_t]] \\
&= \int f(y)\,\mathbb{E}^{\mathbb{W}_x}[G|\omega_t = y]\,e^{-v(t,x,y)}\,dy \\
&= \int_0^\infty f(y)\,\mathbb{E}^{\mathbb{W}_x}[G|\omega_t = y]\,e^{-v(t,x,y)+Q(y)}\,\mu(dy)\,,
\end{aligned}$$

because $\mathbb{E}^{\mathbb{W}_x}[G|\omega_t = y] = 0$ if $y \leq 0$. In other words, the law of X_t restricted to non-extinction has a density with respect to μ given by

$$r(t,x,y) = \mathbb{E}^{\mathbb{W}_x}[G|\omega_t = y]\,e^{-v(t,x,y)+Q(y)}\,.$$

Hence,

$$\begin{aligned}
\int_0^\infty r^2(t,x,y)\,\mu(dy) &= \int \left(\mathbb{E}^{\mathbb{W}_x}[G|\omega_t = y]\,e^{-v(t,x,y)+Q(y)}\right)^2\,e^{-Q(y)+v(t,x,y)}\,e^{-v(t,x,y)}\,dy \\
&= \mathbb{E}^{\mathbb{W}_x}\left[e^{-v(t,x,\omega_t)+Q(\omega_t)}\,\left(\mathbb{E}^{\mathbb{W}_x}[G|\omega_t]\right)^2\right] \\
&\leq \mathbb{E}^{\mathbb{W}_x}\left[e^{-v(t,x,\omega_t)+Q(\omega_t)}\,\mathbb{E}^{\mathbb{W}_x}[G^2|\omega_t]\right] \\
&\leq e^{Q(x)}\,\mathbb{E}^{\mathbb{W}_x}\left[\mathbb{I}_{t<T_0(\omega)}\,e^{-v(t,x,\omega_t)}\,e^{-\int_0^t (q^2-q')(\omega_s)ds}\right]\,,
\end{aligned}$$

where we have used Cauchy-Schwarz's inequality. Since $e^{-v(t,x,.)} \leq (1/2\pi t)^{\frac{1}{2}}$ the proof is completed. □

Remark 2.4. It is interesting to discuss a little bit the conditions we have introduced.

(1) Since q is assumed to be regular, the condition $q^2 - q'$ bounded from below has to be checked only near infinity or near 0.
(2) Consider the behavior near infinity. Let us show that if $\liminf_{y\to\infty}(q^2(y) - q'(y)) = -\infty$ then $\limsup_{y\to\infty}(q^2(y) - q'(y)) > -\infty$ i.e. the drift q is strongly oscillating. Indeed, assume that $q^2(y) - q'(y) \to -\infty$ as $y \to \infty$. It follows that $q'(y) \to \infty$, hence $q(y) \to \infty$. For y large enough we may thus write $q(y) = e^{u(y)}$ for some u going to infinity at infinity. So $e^{2u(y)}(1 - u'(y)e^{-u(y)}) \to -\infty$ implying that $u'e^{-u} \geq 1$ near infinity. Thus if $g = e^{-u}$ we have $g' \leq -1$ i.e. $g(y) \to -\infty$ as $y \to \infty$ which is impossible since g is nonnegative.

(3) If X is given by (1.4) we have

$$q(y) = \frac{1}{y}\left(\frac{1}{2} - \frac{2}{\gamma}h\left(\frac{\gamma y^2}{4}\right)\right).$$

Hence, since h is of class C^1 and $h(0) = 0$, $q^2(y) - q'(y)$ behaves near 0 like $\frac{3}{4y^2}$ so that $q^2 - q'$ is bounded from below near 0 (see Appendix for further conditions fulfilled by h to get the same result near ∞). ♢

3. $\mathbb{L}^2$ and spectral theory of the diffusion process

Theorem 2.3 shows that for a large family of initial laws, the distribution of X_t before extinction has a density belonging to $\mathbb{L}^2(\mu)$. The measure μ is natural since the kernel of the killed process is symmetric in $\mathbb{L}^2(\mu)$, which allows us to use spectral theory.

Let $C_0^\infty((0,\infty))$ be the vector space of infinitely differentiable functions on $(0,\infty)$ with compact support. We denote

$$\langle f, g\rangle_\mu = \int_0^\infty f(u)g(u)\mu(du).$$

Consider the symmetric form

$$\mathcal{E}(f,g) = \langle f', g'\rangle_\mu \quad , \quad D(\mathcal{E}) = C_0^\infty((0,\infty)). \tag{3.1}$$

This form is Markovian and closable. The proof of the latter assertion is similar to the one of Theorem 2.1.4 in [11] just replacing the real line by the positive half line. Its smallest closed extension, again denoted by $\mathcal{E}$, is thus a Dirichlet form which is actually regular and local. According to the theory of Dirichlet forms (see [11] or [12]) we thus know that

- there exists a non-positive self adjoint operator L on $\mathbb{L}^2(\mu)$ with domain $D(L) \supseteq C_0^\infty((0,\infty)$ such that for all f and g in $C_0^\infty((0,\infty))$ the following holds (see [11] Theorem 1.3.1)

$$\mathcal{E}(f,g) = -2\int_0^\infty f(u)\, Lg(u)\,\mu(du) = -2\,\langle f, Lg\rangle_\mu. \tag{3.2}$$

  We point out that for $g \in C_0^\infty((0,\infty))$,

$$Lg = \frac{1}{2}g'' - qg'.$$

- L is the generator of a strongly continuous symmetric semigroup of contractions on $\mathbb{L}^2(\mu)$ denoted by $(P_t)_{t\geq 0}$. This semigroup is (sub)-Markovian, i.e. $0 \leq P_t f \leq 1$ μ a.e. if $0 \leq f \leq 1$ (see [11] Theorem 1.4.1).
- There exists a unique μ-symmetric Hunt process with continuous sample paths (i.e. a diffusion process) up to its explosion time τ whose Dirichlet form is $\mathcal{E}$ (see [11] Theorem 6.2.2)

The last assertion implies that, for μ quasi all $x > 0$ (that is, except for a set of zero capacity, see [11] for details), one can find a probability measure $\mathbb{Q}_x$ on $C(\mathbb{R}^+, (0,\infty))$ such that for all $f \in C_0^\infty((0,\infty))$,

$$f(\omega_{t\wedge\tau}) - f(x) - \int_0^{t\wedge\tau} Lf(\omega_s)ds$$

is a local martingale with quadratic variation $\int_0^{t\wedge\tau} |f'|^2(\omega_s)ds$. Due to our hypothesis $q \in C^1(0,\infty)$ we know that this martingale problem admits a unique solution (see for example [15] page 444). On the other hand, using Itô's formula we know that under $\mathbb{P}_x$, the law of $(X_{t\wedge\tau})$ is also a solution to this martingale problem.

The conclusion is that the semigroup P_t and the semigroup induced by the strong Markov process $(X_{t\wedge\tau})$ coincide on the set of smooth and compactly supported functions. Therefore, for all $f \in \mathbb{L}^2(\mu)$ we have that

$$P_t f(x) = \mathbb{E}[f(X_t)\mathbb{1}_{t<\tau}].$$

Let $(E_\lambda : \lambda \geq 0)$ be the spectral family of $-L$. We can restrict ourselves to the case $\lambda \geq 0$ because $-L$ is nonnegative. Then $\forall\, t \geq 0,\ f,\ g \in \mathbb{L}^2(\mu)$,

$$\int P_t f\, g\, d\mu = \int_0^\infty e^{-\lambda t}\, d\langle E_\lambda f, g\rangle_\mu\,. \tag{3.3}$$

We notice that if absorption is sure, that is (H1) holds, this semigroup coincides with the semigroup of X killed at 0, that is $P_t f(x) = \mathbb{E}[f(X_t^x)\, \mathbb{1}_{t<T_0}]$.
Note that for $f \in \mathbb{L}^2(\mu)$ and all closed interval $K \subset (0,\infty)$,

$$\begin{aligned}
\int (P_t f)^2 d\mu &= \int (P_t(f\mathbb{1}_K + f\mathbb{1}_{K^c}))^2 d\mu \\
&\leq 2\int (P_t(f\mathbb{1}_K))^2 d\mu + 2\int (P_t(f\mathbb{1}_{K^c}))^2 d\mu \\
&\leq 2\int (P_t(f\mathbb{1}_K))^2 d\mu + 2\int (f\mathbb{1}_{K^c})^2 d\mu\,.
\end{aligned}$$

We may choose K large enough in order that the second term in the latter sum is bounded by ε. Similarly we may approximate $f\mathbb{1}_K$ in $\mathbb{L}^2(\mu)$ by $\tilde{f}\mathbb{1}_K$ for some continuous and bounded $\tilde{f}$, up to ε (uniformly in t). Now, thanks to (H1) we know that $P_t(\tilde{f}\mathbb{1}_K)(x)$ goes to 0 as t goes to infinity for any x. Since

$$\int (P_t(\tilde{f}\mathbb{1}_K))^2 d\mu = \int_K \tilde{f}\, P_{2t}(\tilde{f}\mathbb{1}_K)\, d\mu\,,$$

we may apply Lebesgue bounded convergence theorem and conclude that $\int (P_t(\tilde{f}\mathbb{1}_K))^2 d\mu \to 0$ as $t \to \infty$. Hence, we have shown that,

$$\forall\, f \in \mathbb{L}^2(\mu) \qquad \int (P_t f)^2 d\mu \;\to\; 0 \;\text{ as }\; t \;\to\; \infty\,. \tag{3.4}$$

Now we shall introduce the main assumption on q for the spectral aspect of the study.

Hypothesis (H2) $C = -\inf_{y\in(0,\infty)} q^2(y) - q'(y) < \infty$ and $\lim_{y\to\infty} q^2(y) - q'(y) = +\infty\,.$ (3.5)

Proposition 3.1. *Under (H2), $|q(x)|$ tends to infinity as $x \to \infty$, and $q^-(x)$ or $q^+(x)$ tend to 0 as $x \downarrow 0$. If in addition (H1) holds then $q(x) \to \infty$, as $x \to \infty$.*

Proof. Since $q^2 - q'$ tends to ∞ as $x \to \infty$, q does not change sign for large x. If q is bounded near infinity we arrive to a contradiction because q' tends to $-\infty$ and therefore q tends to $-\infty$ as well. So q is unbounded. If $\liminf_{x\to\infty} |q(x)| = a < \infty$ then we can construct a sequence $x_n \to \infty$ of local maxima, or local minima of q whose value $|q(x_n)| < a+1$, but then $q^2(x_n) - q'(x_n)$ stays bounded, which is a contradiction.
Now we prove that $q^-(x)$ or $q^+(x)$ tend to 0 as $x \downarrow 0$. In fact, assume there exist an $\epsilon > 0$ and a sequence (x_n) with $0 < x_n \downarrow 0$ such that $q(x_{2n}) = -\epsilon, q(x_{2n+1}) = \epsilon$. Then we can construct another sequence $z_n \downarrow 0$ such that $|q(z_n)| \le \epsilon$ and $q'(z_n) \to \infty$, contradicting (H2).
Finally, assume (H1) holds. If $q(x) \le -1$ for all $x > x_0$ we arrive to a contradiction. Indeed, for all t

$$\mathbb{P}_{x_0+1}(T_0 > t) \ge \mathbb{P}_{x_0+1}(T_{x_0} > t) \ge \mathbb{P}_{x_0+1}(T_{x_0} = \infty) .$$

The assumption $q(x) \le -1$ implies that $X_t \ge B_t + t$ while $t \le T_{x_0}$, and therefore

$$\mathbb{P}_{x_0+1}(T_0 > t) \ge \mathbb{P}_{x_0+1}\left(B_t + t \text{ hits } \infty \text{ before } x_0\right) = \frac{e^{-2(x_0+1)} - e^{2x_0}}{e^{-2\infty} - e^{2x_0}} = 1 - e^{-2}$$

where we have used that $\big(\exp(-2(B_t+t))\big)$ is a martingale. This contradicts (H1) and we have $q(x) \to \infty$ as $x \to \infty$. $\square$

We may now state the following result.

Theorem 3.2. *If (H2) is satisfied, $-L$ has a purely discrete spectrum $0 \le \lambda_1 < \lambda_2 < \ldots$. Furthermore each λ_i ($i \in \mathbb{N}$) is associated to a unique (up to a multiplicative constant) eigenfunction η_i of class $C^2((0,\infty))$, which also satisfies the ODE*

$$\frac{1}{2}\eta_i'' - q\eta_i' = -\lambda_i \eta_i. \tag{3.6}$$

The sequence $(\eta_i)_{i\ge 1}$ is an orthonormal basis of $\mathbb{L}^2(\mu)$, η_1 can be chosen to be strictly positive in $(0,\infty)$.
For $g \in \mathbb{L}^2(\mu)$,

$$P_t g = \sum_{i\in\mathbb{N}} e^{-\lambda_i t}\langle \eta_i, g\rangle_\mu \eta_i \quad \text{in} \quad \mathbb{L}^2(\mu),$$

then for $f, g \in \mathbb{L}^2(\mu)$,

$$\lim_{t\to\infty} e^{\lambda_1 t}\langle g, P_t f\rangle_\mu = \langle \eta_1, f\rangle_\mu \, \langle \eta_1, g\rangle_\mu \, .$$

If, in addition, (H1) holds, then $\lambda_1 > 0$.

Proof. For $f \in \mathbb{L}^2(dx)$, define $\tilde{P}_t(f) = e^{-Q/2}P_t(f\, e^{Q/2})$, which exists in $\mathbb{L}^2(dx)$ since $f\, e^{Q/2} \in \mathbb{L}^2(\mu)$. $(\tilde{P}_t)_{t\ge0}$ is then a strongly continuous semigroup in $\mathbb{L}^2(dx)$, whose generator $\tilde{L}$ coincides on $C_0^\infty((0,\infty))$ with $\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}(q^2 - q')$ since $C_0^2(0,\infty) \subset D(L)$, and $e^{Q/2} \in C^2(0,\infty)$. The spectral theory of such a Schrödinger operator on the line (or the half line) is well known, but here the potential $v = (q^2 - q')/2$ does not necessarily belong to $\mathbb{L}^\infty_{loc}$ near 0 as it is generally assumed. We shall use [3] chapter 2.
First we follow the proof of Theorem 3.1 in [3]. Since we have assumed that v is bounded from below by $-C/2$, we may consider $H = \tilde{L} - (C/2+1)$, i.e. replace v by $v + C/2 + 1 = w \ge 1$,

hence translate the spectrum. Since for $f \in C_0^\infty(0,\infty)$

$$-(Hf,f) := -\int_0^\infty Hf(u)f(u)\,du = \int_0^\infty \big(|f'(u)|^2/2 + w(u)f^2(u)\big)\,du \geq \int_0^\infty f^2(u)\,du, \quad (3.7)$$

H has a bounded inverse operator. Hence the spectrum of H (and then the one of $\tilde{L}$) will be discrete as soon as H^{-1} is a compact operator, i.e. as soon as $M = \{f \in D(H)\,;\, -(Hf,f) \leq 1\}$ is relatively compact. This is shown in [3] when w is locally bounded, in particular bounded near 0. If w goes to infinity at 0, the situation is even better since our set M is included into the corresponding one with $w \approx 1$ near the origin, which is relatively compact thanks to the asymptotic behavior of v. The conclusion of Theorem 3.1 in [3] is thus still true in our situation, i.e. the spectrum is discrete.

The discussion in Section 2.3 of [3], pp. 59-69, is only concerned with the asymptotic behavior (near infinity) of the solutions of $f'' - 2wf = 0$. Nevertheless, the results there applies to our case. All eigenvalues of $\tilde{L}$ are thus simple (Proposition 3.3 in [3]), and of course the corresponding set of normalized eigenfunctions $(\psi_k)_{k\geq 1}$ is an orthonormal basis of $\mathbb{L}^2(dx)$.

The system $(e^{Q/2}\,\psi_k)_{k\geq 1}$ is thus an orthonormal basis of $\mathbb{L}^2(\mu)$, each $\eta_k = e^{Q/2}\,\psi_k$ being an eigenfunction of L. We can choose them to be $C^2((0,\infty))$ and they satisfy (3.6).

For every $t > 0$, and for every $g, f \in \mathbb{L}^2(\mu)$ we have

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \langle \eta_k, g\rangle_\mu \langle \eta_k, f\rangle_\mu = \langle g, P_t f\rangle_\mu \,.$$

In addition if g and f are nonnegative we get

$$0 \leq \lim_{t\to\infty} e^{\lambda_1 t}\langle g, P_t f\rangle_\mu = \langle \eta_1, f\rangle_\mu \,\langle \eta_1, g\rangle_\mu \,,$$

since $\lambda_1 < \lambda_2 \leq \ldots$ and the sum $\sum_{k=1}^\infty |\langle \eta_k, g\rangle_\mu \langle \eta_k, f\rangle_\mu|$ is finite. It follows that $\langle \eta_1, f\rangle_\mu$ and $\langle \eta_1, g\rangle_\mu$ have the same sign. Changing η_1 into $-\eta_1$ if necessary, we may assume that $\langle \eta_1, f\rangle_\mu \geq 0$ for any nonnegative f, hence $\eta_1 \geq 0$. Since $P_t\eta_1(x) = e^{-\lambda_1 t}\eta_1(x)$ and η_1 is continuous and not trivial, we deduce that $\eta_1(x) > 0$ for all $x > 0$.

Since L is non-positive, $\lambda_1 \geq 0$. Now assume that (H1) holds. Using (3.4) we get for $g \in \mathbb{L}^2(\mu)$

$$0 = \lim_{t\to\infty} \langle P_t g, P_t g\rangle_\mu = \lim_{t\to\infty} e^{-2\lambda_1 t}\langle g, \eta_1\rangle_\mu^2 \,,$$

showing that $\lambda_1 > 0$. □

Moreover, we are able to obtain a pointwise representation of the density r.

Proposition 3.3. *Under (H1) and (H2) we have*

$$r(t,x,y) = \sum_{k=1}^{\infty} e^{-\lambda_k t}\eta_k(x)\eta_k(y), \quad (3.8)$$

uniformly on compact sets of $(0,\infty)\times(0,\infty)\times(0,\infty)$.

Therefore on compact sets of $(0,\infty)\times(0,\infty)$ *we get*

$$\lim_{t\to\infty} e^{\lambda_1 t} r(t,x,y) = \eta_1(x)\eta_1(y). \quad (3.9)$$

Proof. Using Theorems 2.3 and 3.2, for every smooth function g compactly supported on $(0,\infty)$ we have

$$\sum_{k=1}^{n} e^{-\lambda_k t}\langle \eta_k, g\rangle_\mu^2 \le \sum_{k=1}^{\infty} e^{-\lambda_k t}\langle \eta_k, g\rangle_\mu^2 = \iint g(x)g(y)r(t,x,y)e^{-Q(x)-Q(y)}dxdy.$$

Then using the regularity of η_k and r we obtain, by letting $g(y)dy$ tend to the Dirac measure in x, that

$$\sum_{k=1}^{n} e^{-\lambda_k t}\eta_k(x)^2 \le r(t,x,x).$$

Thus, the series $\sum\limits_{k=1}^{\infty} e^{-\lambda_k t}\eta_k(x)^2$ converges pointwise, which by Cauchy-Schwarz inequality implies the pointwise absolute convergence of $\zeta(t,x,y) := \sum\limits_{k=1}^{\infty} e^{-\lambda_k t}\eta_k(x)\eta_k(y)$ and the bound for all n

$$\sum_{k=1}^{n} e^{-\lambda_k t}|\eta_k(x)\eta_k(y)| \le \sqrt{r(t,x,x)}\sqrt{r(t,y,y)}.$$

Using Harnack inequality (see for example [20]) we get

$$\sqrt{r(t,x,x)}\sqrt{r(t,y,y)} \le C_K r(t,x,y)$$

for any x and y in the compact subset K of $(0,\infty)$. Using the dominated convergence theorem we obtain that for all Borel functions g, f with compact support in $(0,\infty)$

$$\int\int g(x)f(y)\zeta(t,x,y)e^{-Q(x)-Q(y)}dxdy = \int\int g(x)f(y)r(t,x,y)e^{-Q(x)-Q(y)}dxdy.$$

Therefore $\zeta(t,x,y) = r(t,x,y)$ $dxdy$-a.s., which proves the almost sure version of (3.8).

Since η_k are smooth eigenfunctions we get the pointwise equality

$$\begin{aligned} e^{-\lambda_k t}\eta_k(x)^2 &= e^{-\lambda_k t/3}\langle r(t/3,x,\bullet),\eta_k\rangle_\mu \langle r(t/3,x,\bullet),\eta_k\rangle_\mu \\ &= \int\int r(t/3,x,y)r(t/3,x,z)e^{-\lambda_k t/3}\eta_k(y)\eta_k(z)e^{-Q(z)-Q(y)}dydz, \end{aligned}$$

which together with the fact $r(t/3,x,\bullet)\in\mathbb{L}^2(\mu)$ and Theorem 2.3 allow us to deduce

$$\begin{aligned} \textstyle\sum_{k=1}^{\infty} e^{-\lambda_k t}\eta_k(x)^2 &= \textstyle\int\int r(t/3,x,y)r(t/3,x,z)\sum_{k=1}^{\infty} e^{-\lambda_k t/3}\eta_k(y)\eta_k(z)e^{-Q(z)-Q(y)}dydz \\ &= \textstyle\int\int r(t/3,x,y)r(t/3,x,z)r(t/3,y,z)e^{-Q(z)-Q(y)}dydz = r(t,x,x). \end{aligned}$$

Dini's theorem then proves the uniform convergence in compacts of $(0,\infty)$ for the series

$$\sum_{k=1}^{\infty} e^{-\lambda_k t}\eta_k(x)^2 = r(t,x,x).$$

By the Cauchy-Schwarz inequality we have for any n

$$\left|\sum_{k=n}^{\infty} e^{-\lambda_k t}\eta_k(x)\eta_k(y)\right| \le \left(\sum_{k=n}^{\infty} e^{-\lambda_k t}\eta_k(x)^2\right)^{1/2}\left(\sum_{k=n}^{\infty} e^{-\lambda_k t}\eta_k(y)^2\right)^{1/2}.$$

This together with the dominated convergence theorem yields (3.9). □

In the previous theorem, notice that $\sum_k e^{-\lambda_k t} = \int r(t,x,x)e^{-Q(x)}dx$ is the $\mathbb{L}^1(\mu)$ norm of $x\mapsto r(t,x,x)$. This is finite if and only if P_t is a trace-class operator on $\mathbb{L}^2(\mu)$.

4. Properties of the eigenfunctions

In this section, we study some properties of the eigenfunctions η_i, including their integrability with respect to μ.

Proposition 4.1. *Assume that (H1) and (H2) are satisfied. Then* $\int_1^\infty \eta_1 e^{-Q} dx < \infty$*,* $F(x) = \eta_1'(x)e^{-Q(x)}$ *is a nonnegative decreasing function and the following limits exist*

$$F(0^+) = \lim_{x \downarrow 0} \eta_1'(x)e^{-Q(x)} \in (0, \infty], \quad F(\infty) = \lim_{x \to \infty} \eta_1'(x)e^{-Q(x)} \in [0, \infty).$$

Moreover $\int\limits_0^\infty \eta_1(x)e^{-Q(x)}dx = \frac{F(0^+)-F(\infty)}{2\lambda_1}$*. In particular*

$$\eta_1 \in \mathbb{L}^1(\mu) \text{ if and only if } F(0^+) < \infty.$$

The function η_1 *is increasing and* $\int\limits_1^\infty e^{-Q(y)}dy < \infty$*.*

Remark 4.2. Note that $g = \eta_1 e^{-Q}$ satisfies the adjoint equation $\frac{1}{2}g'' + (qg)' = -\lambda_1 g$, and then $F(x) = g'(x) + 2q(x)g(x)$ represents the flux at x. Then $\eta_1 \in \mathbb{L}^1(\mu)$ or equivalently $g \in \mathbb{L}^1(dx)$ if and only if the flux at 0 is finite. ♢

Proof. Since η_1 satisfies $\eta_1''(x) - 2q\eta_1'(x) = -2\lambda_1\eta_1(x)$, we obtain for x_0 and x in $(0, \infty)$

$$\eta_1'(x)e^{-Q(x)} = \eta_1'(x_0)e^{-Q(x_0)} - 2\lambda_1 \int_{x_0}^{x} \eta_1(y)e^{-Q(y)}dy, \tag{4.1}$$

and $F = \eta_1' e^{-Q}$ is decreasing. Integrating further gives

$$\eta_1(x) = \eta_1(x_0) + \int_{x_0}^{x} \left(\eta_1'(x_0)e^{-Q(x_0)} - 2\lambda_1 \int_{x_0}^{z} \eta_1(y)e^{-Q(y)}dy \right) e^{Q(z)}dz.$$

If for some $z_0 > x_0$ it holds that $\eta_1'(x_0)e^{-Q(x_0)} - 2\lambda_1 \int_{x_0}^{z_0} \eta_1(y)e^{-Q(y)}dy < 0$, then this inequality holds for all $z > z_0$ since the quantity

$$\eta_1'(x_0)e^{-Q(x_0)} - 2\lambda_1 \int_{x_0}^{z} \eta_1(y)e^{-Q(y)}dy$$

is decreasing in z. This implies that for large x the function η_1 is negative, because $e^{Q(z)}$ tends to ∞ as $z \to \infty$. This is a contradiction and we deduce that for all $x > 0$

$$2\lambda_1 \int_{x}^{\infty} \eta_1(y)e^{-Q(y)}dy \le \eta_1'(x)e^{-Q(x)}.$$

This implies that η_1 is increasing and, being nonnegative, it is bounded near 0. In particular, $\eta_1(0^+)$ exists. Also we deduce that $F \ge 0$ and that $\int\limits_1^\infty e^{-Q(y)}dy < \infty$. We can take the limit as $x \to \infty$ in (4.1) to get

$$F(\infty) = \lim_{x \to \infty} \eta_1'(x)e^{-Q(x)} \in [0, \infty),$$

and $\eta_1'(x_0)e^{-Q(x_0)} = F(\infty) + 2\lambda_1 \int_{x_0}^\infty \eta_1(y)e^{-Q(y)}dy$. From this equality the result follows. □

In the next results we give some sufficient conditions, in terms of q, for the integrability of the eigenfunctions. A first useful condition is the following one

Hypothesis (H3): $\displaystyle\int_0^1 \frac{1}{q^2(y) - q'(y) + C + 2}\, e^{-Q(y)} dy < \infty,$

where as before $C = -\inf_{x>0}(q^2(x) - q'(x))$.

Proposition 4.3. *Assume that (H1), (H2) and (H3) are satisfied. Then η_i belongs to $\mathbb{L}^1(\mu)$ for all i.*

Proof. Recall that $\psi_i = e^{-Q/2}\eta_i$ is an eigenfunction of the Schrödinger operator H introduced in the proof of Theorem 3.2. Replacing f by ψ_i in (3.7) thus yields

$$(C/2 + 1 + \lambda_i) \int_0^\infty \psi_i^2(y)dy = \int_0^\infty (|\psi_i'|^2(y)/2 + w(y)\psi_i^2(y))dy\ .$$

Since the left hand side is finite, the right hand side is finite, in particular

$$\int_0^\infty w(y)\eta_i^2(y)\mu(dy) \ = \ \int_0^\infty w(y)\psi_i^2(y)dy < \infty\,.$$

As a consequence, using Cauchy-Schwarz inequality we get on one hand

$$\int_0^1 |\eta_i(y)|\mu(dy) \ \le \ \left(\int_0^1 w(y)\,\eta_i^2(y)\mu(dy)\right)^{\frac12} \left(\int_0^1 \frac{1}{w(y)}\,\mu(dy)\right)^{\frac12} < \infty$$

thanks to (H3). On the other hand

$$\int_1^\infty |\eta_i(y)|\mu(dy) \ \le \ \left(\int_1^\infty \eta_i^2(y)\mu(dy)\right)^{\frac12} \left(\int_1^\infty \mu(dy)\right)^{\frac12} < \infty$$

according to Proposition 4.1. We have thus proved that $\eta_i \in \mathbb{L}^1(\mu)$. □

We now obtain sharper estimates using properties of the Dirichlet heat kernel. For this reason we introduce

Hypothesis (H4): $\displaystyle\int_1^\infty e^{-Q(x)}dx < \infty \quad \text{and} \quad \int_0^1 x\, e^{-Q(x)/2}dx < \infty.$

Proposition 4.4. *Assume (H2) and (H4) hold. Then all eigenfunctions η_k belong to $\mathbb{L}^1(\mu)$, and there is a constant $K_1 > 0$ such that for any $x \in (0,\infty)$ and any k*

$$|\eta_k(x)| \le K_1\, e^{\lambda_k}\, e^{Q(x)/2}.$$

Moreover η_1 is strictly positive on $\mathbb{R}^+$, and there is a constant $K_2 > 0$ such that for any $x \in (0,1]$ and any k

$$|\eta_k(x)| \le K_2\, x\, e^{2\lambda_k}\, e^{Q(x)/2}.$$

Proof. In Section 3 we introduced the semigroup $\tilde{P}_t$ associated with the Schrödinger equation and showed that $\eta_k = e^{\frac{Q}{2}}\psi_k$, where ψ_k is the unique eigenfunction related to the eigenvalue λ_k for $\tilde{P}_t$. Using estimates on this semigroup, we will get some properties of ψ_k, and we will prove the proposition.

The semigroup $\tilde{P}_t$ is given for $f \in \mathbb{L}^2\big(\mathbb{R}^+, dx\big)$ by

$$\tilde{P}_t f(x) = \mathbb{E}^{\mathbb{W}_x}\left[f(\omega(t))\, \mathbb{1}_{t<T_0}\, \exp\left(-\frac{1}{2}\int_0^t (q^2 - q')(\omega_s)ds\right)\right],$$

where $\mathbb{E}^{\mathbb{W}_x}$ denotes the expectation w.r.t. the Wiener measure starting from x. We first establish a basic estimate on its kernel $\tilde{p}_t(x, y)$.

Lemma 4.5. *Assume condition (H2) holds. There exists a constant $K_3 > 0$ and a continuous increasing function B defined on $[0, \infty)$ satisfying $\lim_{z\to\infty} B(z) = \infty$, such that for any $x > 0, y > 0$ we have*

$$0 < \tilde{p}_1(x, y) \le e^{-(x-y)^2/4} e^{-B(\max\{x,y\})}. \tag{4.2}$$

and

$$\tilde{p}_1(x, y) \le K_3\, p_1^D(x, y), \tag{4.3}$$

where p_t^D is the Dirichlet heat kernel in $\mathbb{R}^+$ given for $x, y \in \mathbb{R}^+$ by

$$p_t^D(x, y) = \frac{1}{\sqrt{2\pi t}}\left(e^{\frac{-(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}}\right).$$

The proof of this lemma is postponed to the Appendix.

It follows immediately from the previous lemma that the kernel $\tilde{p}_1(x, y)$ defines a bounded operator $\tilde{P}_1$ from $\mathbb{L}^2\big(\mathbb{R}^+, dx\big)$ to $\mathbb{L}^\infty\big(\mathbb{R}^+, dx\big)$. As a byproduct, we get that all eigenfunctions ψ_k of $\tilde{P}_1$ are bounded, and more precisely

$$|\psi_k| \le K_1\, e^{\lambda_k}.$$

One also deduces from the previous lemma that the kernel defined for $M > 0$ by

$$\tilde{p}_1^M(x, y) = \mathbb{1}_{x<M}\mathbb{1}_{y<M}\, \tilde{p}_1(x, y)$$

is a Hilbert-Schmidt operator in $\mathbb{L}^2\big(\mathbb{R}^+, dx\big)$, in particular is a compact operator (see for example [7, pages 177, 267]). In addition, it follows at once again from Lemma 4.5 that if $\tilde{P}_1^M$ denotes the operator with kernel $\tilde{p}_1^M$, we have the following estimate, in the norm of operators acting on $\mathbb{L}^2\big(\mathbb{R}^+, dx\big)$,

$$\big\|\tilde{P}_1^M - \tilde{P}_1\big\|_{\mathbb{L}^2(\mathbb{R}^+, dx)} \le C' e^{-B(M)}$$

where C' is a positive constant independent of M. Since $\lim_{M\to\infty} B(M) = \infty$, the operator $\tilde{P}_1$ is a limit in norm of compact operators in $\mathbb{L}^2\big(\mathbb{R}^+, dx\big)$ and hence compact. Since $\tilde{p}_1(x, y) > 0$, the operator $\tilde{P}_1$ is positivity improving (that is if $0 \ne f \ge 0$ then $\tilde{P}_1 f > 0$) implying that the eigenvector ψ_1 is positive.

We now claim that $|\psi_k(x)| \le K_2\, x\, e^{2\lambda_k}$ for $0 < x \le 1$. We have from Lemma 4.5 and the explicit expression for $p_1^D(x, y)$ the existence of a constant K_3 such that

$$\left|e^{-\lambda_k}\psi_k(x)\right| \le K_3 \int_0^\infty p_1^D(x, y)\, |\psi_k(y)|\, dy \le K_3 \|\psi_k\|_\infty \sqrt{\frac{2}{\pi}} e^{-x^2/2} \int_0^\infty e^{-y^2/2} \sinh(xy)\, dy.$$

We now estimate the integral in the right hand side. Using the convexity property of sinh we get $\sinh(xy) \le x \sinh(y) \le \frac{x}{2} e^y$, for $x \in [0,1], y \ge 0$ which yields

$$\int_0^\infty e^{-y^2/2} \sinh(xy)\, dy \le \frac{x}{2} \int_0^\infty e^{-y^2/2} e^y\, dy$$

proving the claim. Together with hypothesis (H4), this estimate implies that η_k belongs to $\mathbb{L}^1\big((0,1), d\mu\big)$.

Since

$$\eta_k(x) = \psi_k(x)\, e^{Q(x)/2},$$

we have

$$\int_1^\infty \eta_k d\mu = \int_1^\infty \psi_k(x)\, e^{-Q(x)/2} dx$$

which implies $\eta_k \in \mathbb{L}^1\big((1,\infty), d\mu\big)$ using Cauchy-Schwarz's inequality. This finishes the proof of Proposition 4.4. □

Remark 4.6. Let us discuss some easy facts about the hypotheses introduced.

(1) If q and q' extend continuously up to 0, hypotheses (H2), (H3) and (H4) reduce to their counterpart at infinity.

(2) Consider $q(x) = \frac{a}{x} + g(x)$ with g a C^1 function up to 0. In order that (2.3) holds at the origin we need $a > -\frac{1}{2}$. Then $\mu(dx) = \Theta(x) x^{-2a} dx$ with Θ bounded near the origin, while $q^2(x) - q'(x) \approx (a + a^2)/x^2$. Hence for (H2) to hold, we need $a \ge 0$. Now we have the estimates

$$\int_0^\varepsilon \frac{1}{(q^2(x) - q'(x) + C + 2)}\, \mu(dx) \approx \int_0^\varepsilon \Theta(x) x^{2(1-a)} dx\,,$$

and

$$\int_0^\varepsilon x e^{-\frac{Q(x)}{2}} dx \approx \int_0^\varepsilon \Theta(x) x^{1-a} dx\,.$$

Therefore (H3) holds for $a < \frac{3}{2}$ and (H4) holds (at 0) for $a < 2$. The conclusion is that $a \in [0, \frac{3}{2})$.

We recall that $a = \frac{1}{2}$ if X comes from a generalized Feller diffusion.

(3) If $q(x) \ge 0$ for x large, hypothesis (H2) implies the first part of hypothesis (H4). Indeed, take $a > 0$ be such that for any $x \ge a$ we have $q(x) > 0$ and $q^2(x) - q'(x) > 1$. Consider the function $y = e^{-Q/2}$ which satisfies $y' = -qy$ and $y'' = (q^2 - q')y$. For $b > a$ we get after integration by parts

$$0 = \int_a^b \left((q^2 - q')y^2 - yy''\right) dx = \int_a^b \left((q^2 - q')y^2 + {y'}^2\right) dx - y(b)y'(b) + y(a)y'(a)\ .$$

Using $y' = -qy$ we obtain

$$\int_a^b y^2 dx \le \int_a^b \left((q^2 - q')y^2 + {y'}^2\right) dx = q(a)y(a)^2 - q(b)y(b)^2 \le q(a)y(a)^2 < \infty$$

and the result follows by letting b tend to infinity.

♢

5. Quasi-stationary distribution and Yaglom limit

Existence of the Yaglom limit and of *q.s.d.* for killed one-dimensional diffusion processes have already been proved by various authors, following the pioneering work by Mandl [26] (see e.g. [5, 27, 35] and references therein). One of the main assumptions in these papers is $\kappa(\infty) = \infty$ and

$$\int_1^\infty e^{-Q(y)} \left(\int_1^y e^{Q(z)}\, dz\right) dy \;=\; +\infty$$

which is not necessarily satisfied in our case. Indeed, under mild conditions, the Laplace method yields that $\int_1^y e^{Q(z)}\, dz$ behaves like $e^{Q(y)}/2q(y)$ when y tends to infinity, so the above equality will not hold if q grows too fast to infinity at infinity. Actually, we will be particularly interested in these cases (our forthcoming assumption (H5)), since they are exactly those when the diffusion "comes down from infinity", which ensures uniqueness of the *q.s.d.*. The second assumption in the aforementioned papers is that q is C^1 up to the origin which is not true in our case of interest.

It is useful to introduce the following condition

Definition 5.1. *We say that* ***Hypothesis (H)*** *is satisfied if (H1) and (H2) hold, and moreover $\eta_1 \in \mathbb{L}^1(\mu)$ (which is the case for example under (H3) or (H4)).*

We now study the existence of *q.s.d.* and Yaglom limit in our framework. When $\eta_1 \in \mathbb{L}^1(\mu)$, a natural candidate for being a *q.s.d.* is the normalized measure $\eta_1\mu/\langle\eta_1, 1\rangle_\mu$, which turns out to be the conditional limit distribution.

Theorem 5.2. *Assume that Hypothesis (H) holds. Then*

$$d\nu_1 = \frac{\eta_1 d\mu}{\langle\eta_1, 1\rangle_\mu}$$

is a quasi-stationary distribution, that is for every $t \geq 0$ and any Borel subset A of $(0, \infty)$,

$$\mathbb{P}_{\nu_1}(X_t \in A \,|\, T_0 > t) = \nu_1(A)\,.$$

Also for any $x > 0$ and any Borel subset A of $(0, \infty)$,

$$\lim_{t\to\infty} e^{\lambda_1 t}\, \mathbb{P}_x(T_0 > t) \;=\; \eta_1(x)\, \langle\eta_1, 1\rangle_\mu\,, \tag{5.1}$$

$$\lim_{t\to\infty} e^{\lambda_1 t}\, \mathbb{P}_x(X_t \in A\,,\, T_0 > t) \;=\; \nu_1(A)\, \eta_1(x)\langle\eta_1, 1\rangle_\mu\,.$$

This implies since $\eta_1 > 0$ on $(0, \infty)$

$$\lim_{t\to\infty} \mathbb{P}_x(X_t \in A \,|\, T_0 > t) \;=\; \nu_1(A)\,,$$

and the probability measure ν_1 is the Yaglom limit distribution. Moreover, for any probability measure ρ with compact support in $(0, \infty)$ we have

$$\lim_{t\to\infty} e^{\lambda_1 t}\, \mathbb{P}_\rho(T_0 > t) \;=\langle\eta_1, 1\rangle_\mu \int \eta_1(x)\rho(dx)\;; \tag{5.2}$$

$$\lim_{t\to\infty} e^{\lambda_1 t}\, \mathbb{P}_\rho(X_t \in A\,,\, T_0 > t) \;=\; \nu_1(A)\, \langle\eta_1, 1\rangle_\mu \int \eta_1(x)\, \rho(dx)\;; \tag{5.3}$$

$$\lim_{t\to\infty} \mathbb{P}_\rho(X_t \in A \mid T_0 > t) \;=\; \nu_1(A). \tag{5.4}$$

Proof. Thanks to the symmetry of the semigroup, we have for all f in $\mathbb{L}^2(\mu)$,

$$\int P_t f \eta_1 d\mu = \int f P_t \eta_1 d\mu = e^{-\lambda_1 t} \int f \eta_1 d\mu \,.$$

Since $\eta_1 \in \mathbb{L}^1(\mu)$, this equality extends to all bounded f. In particular we may use it with $f = \mathbb{1}_{(0,\infty)}$ and with $f = \mathbb{1}_A$. Noticing that

$$\int P_t(\mathbb{1}_{(0,\infty)})\, \eta_1 d\mu = \mathbb{P}_{\nu_1}(T_0 > t)\langle \eta_1, 1\rangle_\mu$$

and $\int P_t \mathbb{1}_A \eta_1 d\mu = \mathbb{P}_{\nu_1}(X_t \in A\,,\, T_0 > t)\langle \eta_1, 1\rangle_\mu$, we have shown that ν_1 is a *q.s.d.*.
The rest of the proof is divided into two cases. First assume that μ is a bounded measure. Thanks to Theorem 2.3, we know that for any $x > 0$, any set $A \subset (0,\infty)$ such that $\mathbb{1}_A \in \mathbb{L}^2(\mu)$ and for any $t > 1$

$$\begin{aligned} \mathbb{P}_x(X_t \in A\,,\, T_0 > t) &= \int \mathbb{P}_y(X_{t-1} \in A\,,\, T_0 > t-1)\, r(1,x,y)\, \mu(dy) \\ &= \int P_{t-1}(\mathbb{1}_A)(y)\, r(1,x,y)\, \mu(dy) \\ &= \int \mathbb{1}_A(y)\, (P_{t-1} r(1,x,.))(y)\, \mu(dy)\,. \end{aligned}$$

Since both $\mathbb{1}_A$ and $r(1,x,.)$ are in $\mathbb{L}^2(\mu)$ and since (H2) is satisfied, we obtain using Theorem 3.2

$$\lim_{t\to\infty} e^{\lambda_1 (t-1)}\, \mathbb{P}_x(X_t \in A\,,\, T_0 > t) = \langle \mathbb{1}_A, \eta_1\rangle_\mu \, \langle r(1,x,.), \eta_1\rangle_\mu \,. \tag{5.5}$$

Since

$$\int r(1,x,y)\eta_1(y)\mu(dy) = (P_1\eta_1)(x) = e^{-\lambda_1}\eta_1(x)$$

we get that ν_1 is the Yaglom limit.

If μ is not bounded (i.e. $\mathbb{1}_{(0,\infty)} \notin \mathbb{L}^2(\mu)$) we need an additional result to obtain the Yaglom limit.

Lemma 5.3. *Assume $\eta_1 \in \mathbb{L}^1(\mu)$ then for all $x > 0$, there exists a locally bounded function $\Theta(x)$ such that for all $y > 0$ and all $t > 1$,*

$$r(t,x,y) \;\le\; \Theta(x)\, e^{-\lambda_1 t}\, \eta_1(y)\,. \tag{5.6}$$

We postpone the proof of the lemma and indicate how it is used to conclude the proof of the theorem.
If (5.6) holds, for $t > 1$, $e^{\lambda_1 t}\, r(t,x,.) \in \mathbb{L}^1(\mu)$ and is dominated by $\Theta(x)\,\eta_1$. Since $r(1,x,\cdot) \in \mathbb{L}^2(\mu)$ by Theorem 2.3, using Theorem 3.2 and writing again $r(t,x,.) = P_{t-1} r(1,x,.)$ μ a.s., we deduce that $\lim_{t\to\infty} e^{\lambda_1 t}\, r(t,x,.)$ exists in $\mathbb{L}^2(\mu)$ and is equal to

$$e^{\lambda_1}\langle r(1,x,.), \eta_1\rangle_\mu\, \eta_1(.) = \eta_1(x)\, \eta_1(.)\,.$$

Recall that convergence in $\mathbb{L}^2$ implies almost sure convergence along subsequences. Therefore, for any sequence $t_n \to \infty$ there exists a subsequence t'_n such that

$$\lim_{n\to\infty} e^{\lambda_1 t'_n}\, r(t'_n, x, y) \;=\; \eta_1(x)\, \eta_1(y) \;\text{ for } \mu\text{-almost all } y > 0\,.$$

Since

$$\mathbb{P}_x(T_0 > t'_n) = \int_0^\infty r(t'_n, x, y)\mu(dy)\,,$$

Lebesgue bounded convergence theorem yields

$$\lim_{n\to\infty} e^{\lambda_1 t'_n}\,\mathbb{P}_x(T_0 > t'_n) = \eta_1(x)\int_0^\infty \eta_1(y)\mu(dy)\,.$$

That is (5.5) holds with $A = (0,\infty)$ for the sequence t'_n. Since the limit does not depend on the subsequence, $\lim_{t\to\infty} e^{\lambda_1 t}\,\mathbb{P}_x(T_0 > t)$ exists and is equal to the previous limit, hence (5.5) is still true. The rest of this part follows as before.

For the last part of the theorem, that is passing from the initial Dirac measures at every fixed $x > 0$ to the compactly supported case, we just use that $\Theta(\bullet)$ is bounded on compact sets included in $(0,\infty)$.

Proof of Lemma 5.3. According to the parabolic Harnack's inequality (see for example [36]), for all $x > 0$, one can find $\Theta_0(x) > 0$, which is locally bounded, such that for all $t > 1$, $y > 0$ and z with $|z - x| \leq \rho(x) = \frac{1}{2} \wedge \frac{x}{4}$

$$r(t,x,y) \;\leq\; \Theta_0(x)\, r(t+1,z,y)\,.$$

It follows that

$$\begin{aligned} r(t,x,y) \;&=\; \frac{\left(\int_{|z-x|\leq\rho(x)} r(t,x,y)\eta_1(z)\mu(dz)\right)}{\left(\int_{|z-x|\leq\rho(x)} \eta_1(z)\mu(dz)\right)} \\ &\leq\; \Theta_0(x)\,\frac{\left(\int_{|z-x|\leq\rho(x)} r(t+1,z,y)\eta_1(z)\mu(dz)\right)}{\left(\int_{|z-x|\leq\rho(x)} \eta_1(z)\mu(dz)\right)} \\ &\leq\; \Theta_0(x)\,\frac{\left(\int r(t+1,z,y)\eta_1(z)\mu(dz)\right)}{\left(\int_{|z-x|\leq\rho(x)} \eta_1(z)\mu(dz)\right)} \\ &\leq\; \Theta_0(x)\,\frac{e^{-\lambda_1(t+1)}\,\eta_1(y)}{\left(\int_{|z-x|\leq\rho(x)} \eta_1(z)\mu(dz)\right)}\,, \end{aligned}$$

since $P_{t+1}\eta_1 = e^{-\lambda_1(t+1)}\eta_1$. But $\Theta_1(x) = \int_{|z-x|\leq\rho(x)} \eta_1(z)\mu(dz) > 0$, otherwise η_1, which is a solution of the linear o.d.e. $\frac{1}{2}g'' - qg' + \lambda_1 g = 0$ on $(0,\infty)$, would vanish on the whole interval $|z - x| \leq \rho(x)$, hence on $(0,\infty)$ according to the uniqueness theorem for linear o.d.e's. The proof of the lemma is thus completed with $\Theta = e^{-\lambda_1}\Theta_0/\Theta_1$. □

The positive real number λ_1 is the natural killing rate of the process. Indeed, the limit (5.1) obtained in Theorem 5.2 shows that for any $x > 0$ and any $t > 0$,

$$\lim_{s\to\infty} \frac{\mathbb{P}_x(T_0 > t+s)}{\mathbb{P}_x(T_0 > s)} \;=\; e^{-\lambda_1 t}\,.$$

Let us also remark that

$$\mathbb{P}_{\nu_1}(T_0 > t) = e^{-\lambda_1 t}.$$

In order to control the speed of convergence to the Yaglom limit, we first establish the following lemma.

Lemma 5.4. *Under conditions (H2) and (H4), the operator P_1 is bounded from $\mathbb{L}^\infty(\mu)$ to $\mathbb{L}^2(\mu)$. Moreover, for any compact subset K of $(0,\infty)$, there is a constant C_K such that for any function $f \in \mathbb{L}^1(\mu)$ with support in K we have*

$$\|P_1 f\|_{\mathbb{L}^2(\mu)} \le C_K \|f\|_{\mathbb{L}^1(\mu)}$$

Proof. Let $g \in \mathbb{L}^\infty(\mu)$, since

$$\big|P_1 g\big| \le P_1\big|g\big| \le \|g\|_{\mathbb{L}^\infty(\mu)} ,$$

we get from (H4)

$$\int_1^\infty \big|P_1 g\big|^2 d\mu \le \|g\|^2_{\mathbb{L}^\infty(\mu)} \int_1^\infty e^{-Q(x)} dx .$$

We now recall that (see Section 3)

$$P_1 g(x) = e^{Q(x)/2} \tilde{P}_1 \left(e^{-Q/2} g\right)(x) .$$

It follows from Lemma 4.5 that uniformly in $x \in (0,1]$ we have (using hypothesis (H4))

$$\left|\tilde{P}_1 \left(e^{-Q/2} g\right)(x)\right| \le \mathcal{O}(1)\ \|g\|_{\mathbb{L}^\infty(\mu)} \int_0^\infty e^{-Q(y)/2} e^{-y^2/4} y\, dy \le \mathcal{O}(1)\ \|g\|_{\mathbb{L}^\infty(\mu)} .$$

This implies

$$\int_0^1 \big|P_1 g\big|^2 d\mu = \int_0^1 \big|\tilde{P}_1 \left(e^{-Q/2} g\right)(x)\big|^2 dx \le \mathcal{O}(1) \|g\|^2_{\mathbb{L}^\infty(\mu)} ,$$

and the first part of the lemma follows. For the second part, we have from the Gaussian bound of Lemma 4.5 that for any $x > 0$ and for any f integrable and with support in K

$$\left|\tilde{P}_1 \left(e^{-Q/2} f\right)(x)\right| \le \mathcal{O}(1) \int_K e^{-Q(y)/2} e^{-(x-y)^2/2} |f(y)|\, dy$$

$$\le \mathcal{O}(1) \sup_{z\in K} e^{Q(z)/2} \sup_{z \in K} e^{-(x-z)^2/2} \int_K e^{-Q(y)} |f(y)|\, dy \le \mathcal{O}(1)\ e^{-x^2/4} \int_K e^{-Q(y)} |f(y)|\, dy$$

since K is compact. This implies

$$\int_0^\infty \big|P_1 f\big|^2 d\mu = \int_0^\infty \big|\tilde{P}_1 \left(e^{-Q/2} f\right)(x)\big|^2 dx \le \mathcal{O}(1)\ \|f\|^2_{\mathbb{L}^1(\mu)} .$$

□

We can now use the spectral decomposition of $r(1, x, .)$ to obtain the following convergence result.

Proposition 5.5. *Under conditions (H2) and (H4), for all $x > 0$ and any measurable subset A of $(0,\infty)$, we have*

$$\lim_{t\to\infty} e^{(\lambda_2-\lambda_1)t} \Big(\mathbb{P}_x(X_t \in A \,|\, T_0 > t) - \nu_1(A)\Big) = \frac{\eta_2(x)}{\eta_1(x)} \left(\frac{\langle 1,\eta_1\rangle_\mu \langle \mathbb{1}_A, \eta_2\rangle_\mu - \langle 1,\eta_2\rangle_\mu \langle \mathbb{1}_A, \eta_1\rangle_\mu}{\langle 1, \eta_1\rangle^2_\mu}\right). \quad (5.7)$$

Proof. Let h be a non negative bounded function, with compact support in $(0,\infty)$. By using the semigroup property, Lemma 5.4 and the spectral decomposition for compact self adjoint semigroups (see Theorem 3.2), we have for any $t>2$,

$$\int \mathbb{P}_x(X_t\in A\,,\,T_0>t)h(x)dx = \langle he^Q\,,\,P_t\mathbb{1}_A\rangle_\mu = \langle P_1\big(he^Q\big)\,,\,P_{(t-2)}P_1\mathbb{1}_A\rangle_\mu$$

$$= \langle P_1\big(he^Q\big)\,,\,\eta_1\rangle_\mu\,\langle \eta_1\,,\,P_1\mathbb{1}_A\rangle_\mu e^{-\lambda_1(t-2)} + \langle P_1\big(he^Q\big)\,,\,\eta_2\rangle_\mu\,\langle \eta_2\,,\,P_1\mathbb{1}_A\rangle_\mu e^{-\lambda_2(t-2)} + R(h,A,t)$$

with

$$|R(h,A,t)| \le \sum_{i\ge 3} e^{-\lambda_i(t-2)}\Big|\langle P_1\big(he^Q\big),\eta_i\rangle_\mu\langle \eta_i,P_1\mathbb{1}_A\rangle_\mu\Big| \le e^{-\lambda_3(t-2)}\|P_1\big(he^Q\big)\|_{\mathbb{L}^2(\mu)}\|P_1\mathbb{1}_A\|_{\mathbb{L}^2(\mu)}\,,$$

due to $\lambda_1<\lambda_2<\lambda_3<\ldots$, the Cauchy-Schwarz inequality and Parseval's identity. Note that since P_1 is symmetric with respect to the scalar product, we have $\langle P_1\big(he^Q\big)\,,\,\eta_1\rangle_\mu = e^{-\lambda_1}\langle he^Q,\eta_1\rangle_\mu$ and similarly for η_2. We also have $\langle \eta_1\,,\,P_1\mathbb{1}_A\rangle_\mu = e^{-\lambda_1}\langle \eta_1\,,\,\mathbb{1}_A\rangle_\mu$ and similarly for η_2. It follows immediately from Lemma 5.4 that for any fixed compact subset K of $(0,\infty)$, any A and any h satisfying the hypothesis of the proposition with support contained in K,

$$|R(h,A,t)| \le \mathcal{O}(1)e^{-\lambda_3(t-2)}\|he^Q\|_{\mathbb{L}^1(\mu)} \le \mathcal{O}(1)e^{-\lambda_3(t-2)}\|h\|_{\mathbb{L}^1(dx)}\,,$$

since h has compact support in $(0,\infty)$. Therefore, letting h tend to a Dirac mass, we obtain that for any compact subset K of $(0,\infty)$, there is a constant D_K such that for any $x\in K$, for any measurable subset A of $(0,\infty)$, and for any $t>2$, we have

$$\Big|\mathbb{P}_x(X_t\in A\,,\,T_0>t) - e^{Q(x)}\eta_1(x)\,\langle \eta_1\,,\,\mathbb{1}_A\rangle_\mu e^{-\lambda_1 t} - e^{Q(x)}\eta_2(x)\,\langle \eta_2\,,\,\mathbb{1}_A\rangle_\mu e^{-\lambda_2 t}\Big| \le D_K\,e^{-\lambda_3 t}\,.$$

The proposition follows at once from

$$\mathbb{P}_x(X_t\in A\,|\,T_0>t) = \frac{\mathbb{P}_x(X_t\in A\,,\,T_0>t)}{\mathbb{P}_x(X_t\in(0,\infty)\,,\,T_0>t)}\,.$$

□

6. The Q-process

As in [5] (Theorem B), we can also describe the law of the process conditioned to be never extinct, usually called the Q-process (also see [22]).

Corollary 6.1. *Assume (H) holds. For all $x>0$ and $s\ge 0$ we have*

$$\lim_{t\to\infty}\mathbb{P}_x(X\in B\mid T_0>t) = \mathbb{Q}_x(B)\ \textit{for all } B \textit{ Borel measurable subsets of } C([0,s]),$$

where $\mathbb{Q}_x$ is the law of a diffusion process on $(0,\infty)$, with transition probability densities (w.r.t. Lebesgue measure) given by

$$q(s,x,y) = e^{\lambda_1 s}\,\frac{\eta_1(y)}{\eta_1(x)}\,r(s,x,y)\,e^{-Q(y)}\,,$$

that is, $\mathbb{Q}_x$ is locally absolutely continuous w.r.t. $\mathbb{P}_x$ and

$$\mathbb{Q}_x(X\in B) = \mathbb{E}_x\left(\mathbb{1}_B(X)\,e^{\lambda_1 s}\,\frac{\eta_1(X_s)}{\eta_1(x)}\,,T_0>s\right)\,.$$

Proof. First check thanks to Fubini's theorem and $\kappa(0^+) < \infty$ in Hypothesis (H1), that $\Lambda(0^+) > -\infty$. We can thus slightly change the notation (for this proof only) and define Λ as $\Lambda(x) = \int_0^x e^{Q(y)}\, dy$. From standard diffusion theory, $(\Lambda(X_{t\wedge T_0}); t \geq 0)$ is a local martingale, from which it is easy to derive that for any $y \geq x \geq 0$, $\mathbb{P}_y(T_x < T_0) = \Lambda(y)/\Lambda(x)$.

Now define $v(t,x) = \frac{\mathbb{P}_x(T_0>t)}{\mathbb{P}_1(T_0>t)}$. As in [5, proof of Theorem B], one can prove for any $x \geq 1$, using the strong Markov property at T_x of the diffusion X starting from 1, that $v(t,x) \leq \Lambda(x)/\Lambda(1)$. On the other hand for $x \leq 1$ we obtain $v(t,x) \leq 1$. Then for any $x \geq 0$ we get $v(t,x) \leq 1 + \Lambda(x)/\Lambda(1)$.

Now thanks to Theorem 5.2, for all x, $e^{\lambda_1 t}\mathbb{P}_x(T_0 > t) \to \eta_1(x)\langle 1, \eta_1\rangle_\mu$ as $t \to \infty$, and

$$\lim_{t\to\infty} v(t,x) = \frac{\eta_1(x)}{\eta_1(1)}\,.$$

Using the Markov property, it is easily seen that for large t,

$$\mathbb{P}_x(X \in B \mid T_0 > t) = \mathbb{E}_x\left[\mathbb{1}_B(X)\, v(t-s, X_s), T_0 > s\right]\, \frac{\mathbb{P}_1(T_0 > t-s)}{\mathbb{P}_x(T_0 > t)}\,.$$

The random variable in the expectation is (positive and) bounded from above by $1 + \Lambda(X_s)/\Lambda(1)$, which is integrable (see below), so we obtain the desired result using Lebesgue bounded convergence theorem.

To see that $\mathbb{E}_x\left(\Lambda(X_s)\, \mathbb{1}_{s<T_0}\right)$ is finite, it is enough to use Itô's formula with the harmonic function Λ up to time $T_0 \wedge T_M$. Since Λ is nonnegative it easily yields $\mathbb{E}_x\left(\Lambda(X_s)\, \mathbb{1}_{s<T_0\wedge T_M}\right) \leq \Lambda(x)$ for all $M > 0$. Letting M go to infinity the indicator converges almost surely to $\mathbb{1}_{s<T_0}$ (thanks to Hypothesis (H1)) so the monotone convergence theorem yields $\mathbb{E}_x\left(\Lambda(X_s)\, \mathbb{1}_{s<T_0}\right) \leq \Lambda(x)$. □

Recall that ν_1 is the Yaglom limit.

Corollary 6.2. *Assume (H) holds. Then for any Borel subset $B \subseteq (0,\infty)$ and any x,*

$$\lim_{s\to\infty} \mathbb{Q}_x(X_s \in B) = \int_B \eta_1^2(y)\mu(dy) \;= \langle \eta_1, 1\rangle_\mu \int_B \eta_1(y)\nu_1(dy).$$

Proof. We know from the proof of Theorem 5.2 that $e^{\lambda_1 s} r(s,x,.)$ converges to $\eta_1(x)\,\eta_1(.)$ in $\mathbb{L}^2(\mu)$ as $s \to \infty$. Hence, since $\mathbb{1}_B\eta_1 \in \mathbb{L}^2(\mu)$,

$$\eta_1(x)\mathbb{Q}_x(X_s \in B) = \int \mathbb{1}_B(y)\eta_1(y)\, e^{\lambda_1 s} r(s,x,y)\mu(dy) \to \eta_1(x) \int_B \eta_1^2(y)\mu(dy)$$

as $s \to \infty$. We remind the reader that $d\nu_1 = \eta_1 d\mu/\langle \eta_1, 1\rangle_\mu$. □

Remark 6.3. The stationary measure of the Q-process is absolutely continuous w.r.t. ν_1, with Radon-Nikodym derivative $\langle \eta_1, 1\rangle_\mu \eta_1$ which thanks to Proposition 4.1 is *nondecreasing*. In particular, the ergodic measure of the Q-process dominates stochastically the Yaglom limit. We refer to [27, 22] for further discussion of the relationship between *q.s.d.* and ergodic measure of the Q-process. ♢

7. **Infinity is an entrance boundary and uniqueness of** *q.s.d.*

We start with the notion of quasi limiting distribution *q.l.d.*.

Definition 7.1. *A probability measure π supported on $(0, \infty)$ is a q.l.d. if there exists a probability measure ν such that the following limit exists in distribution*

$$\lim_{t\to\infty} \mathbb{P}_\nu(X_t \in \bullet \mid T_0 > t) = \pi(\bullet).$$

We also say that ν is attracted to π, or is in the domain of attraction of π, for the conditional evolution.

Obviously every *q.s.d.* is a *q.l.d.*, because such measures are fixed points for the conditional evolution. We prove that the reciprocal is also true, so both concepts coincide.

Lemma 7.2. *Let π a probability measure supported on $(0, \infty)$. If π is a q.l.d. then π is a q.s.d.. In particular there exists $\alpha \geq 0$ such that for all $s > 0$*

$$\mathbb{P}_\pi(T_0 > s) = e^{-\alpha s}.$$

Proof. By hypothesis there exists a probability measure ν such that $\lim_{t\to\infty} \mathbb{P}_\nu(X_t \in \bullet \mid T_0 > t) = \pi(\bullet)$, in distribution. That is, for all continuous and bounded functions f we have

$$\lim_{t\to\infty} \frac{\mathbb{P}_\nu(f(X_t), T_0 > t)}{\mathbb{P}_\nu(T_0 > t)} = \int f(x)\pi(dx).$$

If we take $f(x) = \mathbb{P}_x(X_s \in A, T_0 > s)$, since $f(x) = \int_A r(t, x, y)\mu(dy)$, an application of Harnack's inequality and of the dominated convergence theorem ensures that f is continuous in $(0, \infty)$.

First, take $A = (0, \infty)$, so that $f(x) = \mathbb{P}_x(T_0 > s)$. Then, we obtain for all $s \geq 0$

$$\lim_{t\to\infty} \frac{\mathbb{P}_\nu(T_0 > t + s)}{\mathbb{P}_\nu(T_0 > t)} = \mathbb{P}_\pi(T_0 > s).$$

The left hand side is easily seen to be exponential in s and then there exists $\alpha \geq 0$ such that

$$\mathbb{P}_\pi(T_0 > s) = e^{-\alpha s}.$$

Second, take $f(x) = \mathbb{P}_x(X_s \in A, T_0 > s)$ to conclude that

$$\begin{aligned} \mathbb{P}_\pi(X_s \in A, T_0 > s) &= \lim_{t\to\infty} \mathbb{P}_\nu(f(X_t) | T_0 > t) = \lim_{t\to\infty} \mathbb{P}_\nu(X_{t+s} \in A \mid T_0 > t + s)\tfrac{\mathbb{P}_\nu(T_0 > t+s)}{\mathbb{P}_\nu(T_0 > t)} \\ &= e^{-\alpha s}\pi(A), \end{aligned}$$

and then π is a *q.s.d.*. $\square$

Recall from Theorem 5.2 that under Hypothesis (H), the measure $d\nu_1 = \eta_1 d\mu / \langle \eta_1, 1\rangle_\mu$ is the Yaglom limit, which in addition is a *q.l.d.* attracting all initial distribution with compact support on $(0, \infty)$. It is natural to ask about the uniqueness of the *q.s.d.*. Here again, our assumptions on the behavior of q at infinity will allow us to characterize the domain of attraction of the *q.s.d.* ν_1 associated to η_1. This turns out to be entirely different from the cases studied in [5] for instance.

We say that the diffusion process X *comes down from infinity* if there is $y > 0$ and a time $t > 0$ such that

$$\lim_{x\uparrow\infty} \mathbb{P}_x(T_y < t) \;>\; 0.$$

This terminology is equivalent to the property that ∞ is an entrance boundary for X (for instance see [31, page 283]).

Let us introduce the following condition

Hypothesis (H5): $\displaystyle\int_1^\infty e^{Q(y)}\int_y^\infty e^{-Q(z)}\,dz\,dy\,<\,\infty.$

Tonelli's Theorem ensures that (H5) is equivalent to

$$\int_1^\infty e^{-Q(y)}\int_1^y e^{Q(z)}dz\,dy<\infty. \tag{7.1}$$

If (H5) holds then for $y\geq 1$, $\int_y^\infty e^{-Q(z)}\,dz<\infty$. Applying the Cauchy-Schwarz inequality we get $(x-1)^2=\left(\int_1^x e^{Q/2}e^{-Q/2}dz\right)^2\leq\int_1^x e^Q dz\ \int_1^x e^{-Q}dz$, and therefore (H5) implies that $\Lambda(\infty)=\infty$.

Now we state the main result of this section.

Theorem 7.3. *Assume (H) holds. Then the following are equivalent:*

(i) *X comes down from infinity;*
(ii) *(H5) holds;*
(iii) *ν_1 attracts all initial distributions ν supported in $(0,\infty)$, that is*

$$\lim_{t\to\infty}\mathbb{P}_\nu(X_t\in\bullet\mid T_0>t)=\nu_1(\bullet).$$

In particular any of these three conditions implies that there is a unique q.s.d..

Remark 7.4. It is not obvious when Condition (H5) holds. In this direction, the following explicit conditions on q, all together, are sufficient for (H5) to hold:

- $q(x)\geq q_0>0$ for all $x\geq x_0$
- $\limsup_{x\to\infty}q'(x)/2q^2(x)<1$
- $\int_{x_0}^\infty\frac{1}{q(x)}\,dx<\infty\,.$

Indeed, check first that these conditions imply that $q(x)$ goes to infinity as $x\to\infty$. Then defining $s(y):=\int_y^\infty e^{-Q(z)}\,dz$, the first condition above implies that $s(y)e^{Q(y)}$ is bounded in $y\geq x_0$. Integrating by parts on $\int se^Q dz$ gives

$$\int_{x_0}^x se^Q dz=\int_{x_0}^x\frac{s}{2q}\,e^Q 2q\,dz=\frac{s}{2q}e^Q\Big|_{x_0}^x+\int_{x_0}^x\frac{1}{2q}dz+\int_{x_0}^x se^Q\frac{q'}{2q^2}dz.$$

Since $se^Q/2q$ vanishes at infinity, the third condition implies that $se^Q(1-q'/2q^2)$ is integrable and thanks to the second condition we conclude that (H5) holds.

On the other hand, if (H5) holds, $q'(x)\geq 0$ for $x\geq x_0$ and $q(x_0)>0$, then $q(x)$ goes to infinity as $x\to\infty$ and $\int_{x_0}^\infty\frac{1}{q(x)}\,dx<\infty\,.$

We can retain that under the assumption that $q'(x)\geq 0$ for $x\geq x_0$ and $q(x)$ goes to infinity as $x\to\infty$, then

$$\text{(H5)}\iff\int_1^\infty\frac{1}{q(x)}\,dx<\infty\,.$$

Indeed, the only thing left to prove is the sufficiency of (H5). Since $s(y)$ tends to 0 as $y \to \infty$ (because Q grows at least linearly), then by the mean value theorem we have

$$\frac{\int_y^\infty e^{-Q(z)}dz}{e^{-Q(y)}} = \frac{1}{2q(\xi)},$$

where $\xi \in [y, \infty)$. Using that q is monotone we obtain the bound

$$\frac{\int_y^\infty e^{-Q(z)}dz}{e^{-Q(y)}} \le \frac{1}{2q(y)},$$

and the equivalence is shown. $\diamondsuit$

The proof of Theorem 7.3 follows from Propositions 7.5, 7.6 and 7.7.

Proposition 7.5. *Assume (H1) holds. If there is a unique q.s.d. that attracts all initial distributions supported in* $(0, \infty)$*, then* X *comes down from infinity.*

Proof. Let π be the unique *q.s.d.* that attracts all distributions. We know that $\mathbb{P}_\pi(T_0 > t) = e^{-\alpha t}$ for some $\alpha \ge 0$. Since absorption is certain then $\alpha > 0$. For the rest of the proof let ν be any initial distribution supported on $(0, \infty)$, which by hypothesis is in the domain of attraction of π that is, for any bounded and continuous function f we have

$$\lim_{t\to\infty} \int_0^\infty \mathbb{P}_\nu(X_t \in dx \mid T_0 > t) f(x) = \int_0^\infty f(x)\pi(dx).$$

We now prove that for any $\lambda < \alpha$, $\mathbb{E}_\nu(e^{\lambda T_0}) < \infty$. As in Lemma 7.2 we have for any s

$$\lim_{t\to\infty} \frac{\mathbb{P}_\nu(T_0 > t + s)}{\mathbb{P}_\nu(T_0 > t)} = e^{-\alpha s}\,.$$

Now pick $\lambda \in (0, \alpha)$ and $\varepsilon > 0$ such that $(1 + \varepsilon)e^{\lambda-\alpha} < 1$. An elementary induction shows that there is t_0 such that for any $t > t_0$, and any integer n

$$\frac{\mathbb{P}_\nu(T_0 > t + n)}{\mathbb{P}_\nu(T_0 > t)} \le (1 + \varepsilon)^n e^{-\alpha n}\,.$$

Breaking down the integral $\int_{t_0}^\infty \mathbb{P}_\nu(T_0 > s)\, e^{\lambda s}\, ds$ over intervals of the form $(n, n + 1]$ and using the previous inequality, it is easily seen that this integral converges. This proves that $\mathbb{E}_\nu(e^{\lambda T_0}) < \infty$ for any initial distribution ν.

Now fix $\lambda = \alpha/2$ and for any $x \ge 0$, let $g(x) = \mathbb{E}_x(e^{\lambda T_0}) < \infty$. We want to show that g is bounded, which trivially entails that X comes down from infinity. Thanks to the previous step, for any nonnegative random variable Y with law ν

$$\mathbb{E}(g(Y)) = \mathbb{E}_\nu(e^{\lambda T_0}) < \infty.$$

Since Y can be any random variable, this implies that g is bounded. Indeed, observe that g is increasing and $g(0) = 1$, so that $a := 1/g(\infty)$ is well defined in $[0, 1)$. Then check that

$$\nu(dx) = \frac{g'(x)}{(1 - a)g(x)^2}\, dx$$

is a probability density on $(0,\infty)$. To conclude we use the fact that $\int g\, d\nu < \infty$ to get

$$\int g\, d\nu = \int_0^\infty \frac{g'(x)}{(1-a)g(x)}\, dx = \frac{1}{1-a} \ln g(x)\big|_0^\infty = \frac{\ln g(\infty)}{1-a},$$

and then g is bounded.

□

Proposition 7.6. *The following are equivalent*

(i) *X comes down from infinity;*
(ii) *(H5) holds;*
(iii) *for any $a > 0$ there exists $y_a > 0$ such that* $\sup_{x>y_a} \mathbb{E}_x[e^{aT_{y_a}}] < \infty$.

Proof. Since (i) is equivalent to ∞ being an entrance boundary and (ii) is equivalent to (7.1) we must show that "∞ is an entrance boundary" and (7.1) are equivalent. This will follow from [17, Theorem 20.12,(iii)]. For that purpose consider $Y_t = \Lambda(X_t)$. Under each one of the conditions (i) or (ii) we have $\Lambda(\infty) = \infty$. It is direct to prove that Y is in natural scale on the interval $(\Lambda(0),\infty)$, that is, for $\Lambda(0) < a \le y \le b < \infty = \Lambda(\infty)$

$$\mathbb{P}_y(T_a^Y < T_b^Y) = \frac{b-y}{b-a},$$

where T_a^Y is the hitting time of a for the diffusion Y. Then, ∞ is an entrance boundary for Y if and only if

$$\int_0^\infty y\, m(dy) < \infty,$$

where m is the speed measure of Y, which is given by

$$m(dy) = \frac{2\, dy}{(\Lambda'(\Lambda^{-1}(y)))^2},$$

see [18, formula (5.51)], because Y satisfies the SDE

$$dY_t = \Lambda'(\Lambda^{-1}(Y_t))dB_t.$$

After a change of variables we obtain

$$\int_0^\infty y\, m(dy) = \int_1^\infty e^{-Q(y)} \int_1^x e^{Q(z)} dz\, dx.$$

Therefore we have shown the equivalence between (i) and (ii).
We continue the proof with $(ii) \Rightarrow (iii)$. Let $a > 0$, and pick x_a large enough so that

$$\int_{x_a}^\infty e^{Q(x)} \int_x^\infty e^{-Q(z)}\, dz\, dx \le \frac{1}{2a}.$$

Let J be the nonnegative increasing function defined on $[x_a,\infty)$ by

$$J(x) = \int_{x_a}^x e^{Q(y)} \int_y^\infty e^{-Q(z)}\, dz\, dy.$$

Then check that $J'' = 2qJ' - 1$, so that $LJ = -1/2$. Set now $y_a = 1 + x_a$, and consider a large $M > x$. Itô's formula gives

$$\mathbb{E}_x(e^{a(t\wedge T_M \wedge T_{y_a})}\, J(X_{t\wedge T_M \wedge T_{y_a}})) = J(x) + \mathbb{E}_x\left(\int_0^{t\wedge T_M\wedge T_{y_a}} e^{as}\,(aJ(X_s) + LJ(X_s))\, ds\right).$$

But $LJ = -1/2$, and $J(X_s) < J(\infty) \le 1/(2a)$ for any $s \le T_{y_a}$, so that

$$\mathbb{E}_x[e^{a(t\wedge T_M \wedge T_{y_a})}\, J(X_{t\wedge T_M\wedge T_{y_a}})] \le J(x)\,.$$

But J is increasing, hence for $x \ge y_a$ one gets $1/(2a) > J(x) \ge J(y_a) > 0$. It follows that $\mathbb{E}_x(e^{a(t\wedge T_M\wedge T_{y_a})}) \le 1/(2aJ(y_a))$ and finally $\mathbb{E}_x(e^{aT_{y_a}}) \le 1/(2aJ(y_a))$, by the monotone convergence theorem. So (iii) holds.
Finally, it is clear that $(iii) \Rightarrow (i)$. □

Proposition 7.7. *Assume (H) holds. If there is x_0 such that $\sup_{x\ge x_0} \mathbb{E}_x(e^{\lambda_1 T_{x_0}}) < \infty$, then ν_1 attracts all initial distribution supported in $(0,\infty)$.*

The proof of this result requires the following control near 0 and ∞.

Lemma 7.8. *Assume (H) holds, and $\sup_{x\ge x_0} \mathbb{E}_x(e^{\lambda_1 T_{x_0}}) < \infty$. For $h \in \mathbb{L}^1(\mu)$ strictly positive in $(0,\infty)$ we have*

$$\lim_{\epsilon\downarrow 0}\limsup_{t\to\infty} \frac{\int_0^\epsilon h(x)\mathbb{P}_x(T_0>t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0>t)\mu(dx)} = 0 \tag{7.2}$$

$$\lim_{M\uparrow\infty}\limsup_{t\to\infty} \frac{\int_M^\infty h(x)\mathbb{P}_x(T_0>t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0>t)\mu(dx)} = 0 \tag{7.3}$$

Proof. We start with (7.2). Using Harnack's inequality, we have for $\epsilon < 1$ and large t

$$\frac{\int_0^\epsilon h(x)\mathbb{P}_x(T_0>t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0>t)\mu(dx)} \le \frac{\mathbb{P}_1(T_0>t)\int_0^\epsilon h(z)\mu(dz)}{Cr(t-1,1,1)\int_1^2 h(x)\mu(dx)\int_1^2\mu(dy)},$$

then

$$\limsup_{t\to\infty}\frac{\int_0^\epsilon h(x)\mathbb{P}_x(T_0>t)\mu(dx)}{\int h(x)\mathbb{P}_x(T_0>t)\mu(dx)} \le \limsup_{t\to\infty}\frac{\mathbb{P}_1(T_0>t)\int_0^\epsilon h(z)\mu(dz)}{Cr(t-1,1,1)\int_1^2 h(x)\mu(dx)\int_1^2\mu(dy)}$$

$$= \frac{e^{-\lambda_1}\langle \eta_1, 1\rangle_\mu \int_0^\epsilon h(z)\mu(dz)}{C\,\eta_1(1)\int_1^2 h(x)\mu(dx)\int_1^2\mu(dy)},$$

and the first assertion of the lemma is proven.
For the second limit, we set $A_0 := \sup_{x\ge x_0}\mathbb{E}_x(e^{\lambda_1 T_{x_0}}) < \infty$. Then for large $M > x_0$, we have

$$\mathbb{P}_x(T_0>t) = \int_0^t \mathbb{P}_{x_0}(T_0>u)\mathbb{P}_x(T_{x_0}\in d(t-u)) + \mathbb{P}_x(T_{x_0}>t).$$

Using that $\lim_{u\to\infty} e^{\lambda_1 u}\mathbb{P}_{x_0}(T_0>u) = \eta_1(x_0)\langle\eta_1,1\rangle_\mu$ we obtain that $B_0 := \sup_{u\ge 0} e^{\lambda_1 u}\mathbb{P}_{x_0}(T_0 > u) < \infty$. Then

$$\begin{aligned}\mathbb{P}_x(T_0>t) &\le B_0\int_0^t e^{-\lambda_1 u}\mathbb{P}_x(T_{x_0}\in d(t-u)) + \mathbb{P}_x(T_{x_0}>t)\\ &\le B_0 e^{-\lambda_1 t}\mathbb{E}_x(e^{\lambda_1 T_{x_0}}) + e^{-\lambda_1 t}\mathbb{E}_x(e^{\lambda_1 T_{x_0}}) \le e^{-\lambda_1 t}A_0(B_0+1),\end{aligned}$$

and (7.3) follows immediately. □

Proof of Proposition 7.7. Let ν be any fixed probability distribution whose support is contained in $(0,\infty)$. We must show that the conditional evolution of ν converges to ν_1. We begin by claiming that ν can be assumed to have a strictly positive density h, with respect to μ. Indeed, let

$$\ell(y) = \int r(1,x,y)\nu(dx).$$

Using Tonelli's theorem we have

$$\int\int r(1,x,y)\nu(dx)\,\mu(dy) = \int\int r(1,x,y)\,\mu(dy)\,\nu(dx) = \int \mathbb{P}_x(T_0 > 1)\nu(dx) \le 1,$$

which implies that $\int r(1,x,y)\nu(dx)$ is finite $dy-$a.s.. Also ℓ is strictly positive by Harnack's inequality. Finally, define $h = \ell/\int \ell d\mu$. Notice that for $d\rho = hd\mu$

$$\mathbb{P}_\nu(X_{t+1} \in \bullet \mid T_0 > t+1) = \mathbb{P}_\rho(X_t \in \bullet \mid T_0 > t),$$

showing the claim.

Consider $M > \epsilon > 0$ and any Borel set A included in $(0,\infty)$. Then

$$\left| \frac{\int \mathbb{P}_x(X_t \in A,\, T_0 > t)h(x)\,\mu(dx)}{\int \mathbb{P}_x(T_0 > t)h(x)\,\mu(dx)} - \frac{\int_\epsilon^M \mathbb{P}_x(X_t \in A,\, T_0 > t)h(x)\,\mu(dx)}{\int_\epsilon^M \mathbb{P}_x(T_0 > t)h(x)\,\mu(dx)} \right|$$

is bounded by the sum of the following two terms

$$\begin{aligned} I1 &= \left| \frac{\int \mathbb{P}_x(X_t \in A, T_0 > t)h(x)\,\mu(dx)}{\int \mathbb{P}_x(T_0 > t)h(x)\,\mu(dx)} - \frac{\int_\epsilon^M \mathbb{P}_x(X_t \in A, T_0 > t)h(x)\,\mu(dx)}{\int \mathbb{P}_x(T_0 > t)h(x)\,\mu(dx)} \right| \\ I2 &= \left| \frac{\int_\epsilon^M \mathbb{P}_x(X_t \in A, T_0 > t)h(x)\,\mu(dx)}{\int \mathbb{P}_x(T_0 > t)h(x)\,\mu(dx)} - \frac{\int_\epsilon^M \mathbb{P}_x(X_t \in A, T_0 > t)h(x)\,\mu(dx)}{\int_\epsilon^M \mathbb{P}_x(T_0 > t)h(x)\,\mu(dx)} \right|. \end{aligned}$$

We have the bound

$$I1 \vee I2 \le \frac{\int_0^\epsilon \mathbb{P}_x(T_0 > t)h(x)\,\mu(dx) + \int_M^\infty \mathbb{P}_x(T_0 > t)h(x)\,\mu(dx)}{\int \mathbb{P}_x(T_0 > t)h(x)\,\mu(dx)}.$$

Thus, from Lemma 7.8 we get

$$\lim_{\epsilon\downarrow 0,\, M\uparrow\infty}\ \limsup_{t\to\infty} \left| \frac{\int \mathbb{P}_x(X_t \in A,\, T_0 > t)h(x)\,\mu(dx)}{\int \mathbb{P}_x(T_0 > t)h(x)\,\mu(dx)} - \frac{\int_\epsilon^M \mathbb{P}_x(X_t \in A,\, T_0 > t)h(x)\,\mu(dx)}{\int_\epsilon^M \mathbb{P}_x(T_0 > t)h(x)\,\mu(dx)} \right| = 0.$$

On the other hand we have using (5.4)

$$\lim_{t\to\infty} \frac{\int_\epsilon^M \mathbb{P}_x(X_t \in A,\, T_0 > t)h(x)\,\mu(dx)}{\int_\epsilon^M \mathbb{P}_x(T_0 > t)h(x)\,\mu(dx)} = \frac{\int_A \eta_1(z)\mu(dz)}{\int_{\mathbb{R}^+} \eta_1(z)\mu(dz)} = \nu_1(A),$$

independently of $M > \epsilon > 0$, and the result follows. □

The following corollary of Proposition 7.6 describes how fast the process comes down from infinity.

Corollary 7.9. *Assume (H) and (H5) hold. Then for all $\lambda < \lambda_1$, $\sup_{x>0} \mathbb{E}_x[e^{\lambda T_0}] < \infty$.*

Proof. We have seen in Section 5 (Theorem 5.2) that for all $x > 0$, $\lim_{t\to\infty} e^{\lambda_1 t}\mathbb{P}_x\big(T_0 > t\big) = \eta_1(x)\langle\eta_1, 1\rangle_\mu < \infty$ i.e. $\mathbb{E}_x[e^{\lambda T_0}] < \infty$ for all $\lambda < \lambda_1$. Applying Proposition 7.6 with $a = \lambda$ and the strong Markov property it follows that $\sup_{x>y_\lambda} \mathbb{E}_x[e^{\lambda T_0}] < \infty$. Furthermore, thanks to the uniqueness of the solution of (1.1), $X^x_t \leq X^{y_\lambda}_t$ a.s. for all $t > 0$ and all $x < y_\lambda$, hence $\mathbb{E}_x[e^{\lambda T_0}] \leq \mathbb{E}_{y_\lambda}[e^{\lambda T_0}]$ for those x, completing the proof. □

The previous corollary states that the killing time for the process starting from infinity has exponential moments up to order λ_1. In [21] an explicit calculation of the law of T_0 is done in the case of the logistic Feller diffusion Z (hence the corresponding X) and also for other related models. In particular it is shown in Corollary 3.10 therein, that the absorption time for the process starting from infinity has a finite expectation. As we remarked in studying examples, a very general family of diffusion processes (including the logistic one) satisfy all assumptions in Corollary 7.9, which is thus an improvement of the quoted result.

We end this section by gathering some known results on birth–death processes that are close to our findings. Let Y be a birth–death process with birth rate λ_n and death rate μ_n when in state n. Assume that $\lambda_0 = \mu_0 = 0$ and that extinction (absorption at 0) occurs with probability 1. Let

$$S = \sum_{i\geq 1} \pi_i + \sum_{n\geq 1} (\lambda_n\pi_n)^{-1} \sum_{i\geq n+1} \pi_i\,,$$

where

$$\pi_n = \frac{\lambda_1\lambda_2\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\,.$$

In this context sure absorption at 0, that is (H1), is equivalent to $A := \sum_{i\geq 1}(\lambda_i\pi_i)^{-1} = \infty$ (see [19, formula 7.9]). On the other hand we also have $\mathbb{E}_1(T_0) = \sum_{i\geq 1} \pi_i$. We may state

Proposition 7.10. *For a birth–death process Y that satisfies (H1), the following are equivalent:*

(i) *Y comes down from infinity;*
(ii) *There is one and only one q.s.d.;*
(iii) $\lim_{n\uparrow\infty} \uparrow \mathbb{E}_n(T_0) < \infty$*;*
(iv) $S < \infty$*.*

Proof. In [8, Theorem 3.2], it is stated the following key alternative: $S < \infty$ iff there is a unique *q.s.d.*; if $S = \infty$ then there is no *q.s.d.* or there are infinitely many ones. Then the equivalence between *(ii)* and *(iv)* is immediate. Also it is well known that

$$\mathbb{E}_n(T_0) = \sum_{i\geq 1} \pi_i + \sum_{r=1}^{n-1} (\lambda_r\pi_r)^{-1} \sum_{i\geq r+1} \pi_i\,,$$

see for example [19, formula 7.10]. Therefore (iii) and (iv) are equivalent. Let us now examine how this criterion is related to the nature of the boundary at $+\infty$. From the table in [1, section 8.1] we have that $+\infty$ is an entrance boundary iff $A = \infty$, $\mathbb{E}_1(T_0) < \infty$ and $S < \infty$. Finally, $S < \infty$ implies that $\mathbb{E}_1(T_0) < \infty$ and this ensures $\mathbb{P}_1(T_0 < \infty) = 1$, that is $A = \infty$. This shows the result. □

8. **Biological models**

8.1. **Population dynamics and quasi-stationary distributions.** Our aim is to model the dynamics of an isolated population by a diffusion $Z := (Z_t; t \geq 0)$. Since competition for limited resources impedes natural populations with no immigration to grow indefinitely, they are all doomed to become extinct at some finite time T_0. However, T_0 can be large compared to human timescale and it is common that population sizes fluctuate for large amount of time before extinction actually occurs. The notion of quasi-stationarity captures this behavior [30, 34].

The diffusions we consider arise as scaling limits of general birth–death processes. More precisely, let $(Z^N)_{N\in\mathbb{N}}$ be a sequence of continuous time birth–death processes $Z^N := (Z^N_t; t \geq 0)$, renormalized by the weight N^{-1}, hence taking values in $N^{-1}\mathbb{N}$. Assume that their birth and death rates from state x are equal to $b_N(x)$ and $d_N(x)$, respectively, and $b_N(0) = d_N(0) = 0$, ensuring that the state 0 is absorbing. We also assume that for each N and for some constant B_N,

$$b_N(x) \leq (x+1)B_N, \ x \geq 0$$

and that there exist a nonnegative constant γ and a function $h \in C^1([0,\infty))$, $h(0) = 0$, called the *growth function*, such that

$$\forall x \in (0,\infty): \quad \lim_{N\to\infty} \frac{1}{N}(b_N(x) - d_N(x)) = h(x) \quad ; \quad \lim_{N\to\infty} \frac{1}{2N^2}(b_N(x) + d_N(x)) = \gamma x. \quad (8.1)$$

Important ecological examples include

(i) The *pure branching* case, where the individuals give birth and die independently, so that one can take $b_N(x) = (\gamma N + \lambda)Nx$ and $d_N(x) = (\gamma N + \mu)Nx$. Writing $r := \lambda - \mu$ for the *Malthusian growth parameter* of the population, one gets $h(z) = rz$.

(ii) The *logistic branching* case, where $b_N(x) = (\gamma N + \lambda)Nx$ and $d_N(x) = (\gamma N + \mu)Nx + \frac{c}{N}Nx(Nx-1)$. The quadratic term in the death rate describes the interaction between individuals. The number of individuals is of order N, the biomass of each individual is of order N^{-1}, and c/N is the interaction coefficient. The growth function is then $h(z) = rz - cz^2$.

(iii) Dynamics featuring *Allee effect*, that is, a positive density-dependence for certain ranges of density, corresponding to cooperation in natural populations. A classical type of growth function in that setting is $h(z) = rz(\frac{z}{K_0} - 1)(1 - \frac{z}{K})$. Observe that in this last case, the individual growth rate is no longer a monotone decreasing function of the population size.

Assuming further that $(Z^N_0)_{N\in\mathbb{N}}$ converges as $N \to \infty$ (we thus model the dynamics of a population whose size is of order N), we may prove, following Lipow [24] or using the techniques of Joffe-Métivier [16], that the sequence $(Z^N)_{N\in\mathbb{N}}$ converges weakly to a continuous limit Z. The parameter γ can be interpreted as a demographic parameter describing the ecological timescale. There is a main qualitative difference depending on whether $\gamma = 0$ or not.

If $\gamma = 0$, then the limit Z is a deterministic solution to the dynamical system $\dot{Z}_t = h(Z_t)$. Since $h(0) = 0$, the state 0 is always an equilibrium, but it can be unstable. For example, in the logistic case $h(z) = rz - cz^2$ and it is easily checked that when $r > 0$, the previous dynamical system has two equilibria, 0 which is unstable, and r/c (called *carrying capacity*) which is asymptotically stable. In the Allee effect case, 0 and K are both stable equilibria, but K_0 is an unstable equilibrium, which means the population size has a threshold K_0 to growth, below which it cannot take over.

If $\gamma > 0$, the sequence $(Z^N)_{N\in\mathbb{N}}$ converges in law to the process Z, solution to the following stochastic differential equation

$$dZ_t = \sqrt{\gamma Z_t} dB_t + h(Z_t)dt. \tag{8.2}$$

The acceleration of the ecological process has generated the white noise. Note that $h'(0^+)$ is the mean *per capita* growth rate for *small* populations. The fact that it is finite is mathematically convenient, and biologically reasonable. Since $h(0) = 0$, the population undergoes no immigration, so that 0 is an absorbing state. One can easily check that when time goes to infinity, either Z goes to ∞ or is absorbed at 0.

When $h \equiv 0$, we get the classical Feller diffusion, so we call *generalized Feller diffusions* the diffusions driven by (8.2). When h is linear, we get the general continuous-state branching process with continuous paths, sometimes also called Feller diffusion by extension. When h is concave quadratic, we get the logistic Feller diffusion [9, 21].

Definition 8.1. *(HH) We say that h satisfies the condition (HH) if*

$$(i)\ \lim_{x\to\infty} \frac{h(x)}{\sqrt{x}} = -\infty, \qquad (ii)\ \lim_{x\to\infty} \frac{xh'(x)}{h(x)^2} = 0.$$

In particular (HH) holds for any subcritical branching diffusion, and any logistic Feller diffusion. Concerning assumption (i), the fact that h goes to $-\infty$ indicates strong competition in large populations resulting in negative growth rates (as in the logistic case). Assumption (ii) is fulfilled for most classical biological models, and it appears as a mere technical condition.

Gathering all results of the present paper and applying them to our biological model yields the following statement, which will be proved at the end of this section.

Theorem 8.2. *Let Z be the solution of (8.2). We assume $h \in C^1([0,\infty))$, $h(0) = 0$ and that h satisfies assumption (HH). Then, for all initial laws with bounded support, the law of Z_t conditioned on $\{Z_t \neq 0\}$ converges exponentially fast to a probability measure ν, called the Yaglom limit. The law $\mathbb{Q}_x$ of the process Z starting from x and conditioned to be never extinct exists and defines the so-called Q-process. This process converges, as $t \to \infty$, in distribution, to its unique invariant probability measure. This probability measure is absolutely continuous w.r.t. ν with a nondecreasing Radon–Nikodym derivative.*

In addition, if the following integrability condition is satisfied

$$\int_1^\infty \frac{dx}{-h(x)} < \infty,$$

then Z comes down from infinity and the convergence of the conditional one-dimensional distributions holds for all initial laws. In particular, the Yaglom limit ν is then the unique quasi-stationary distribution.

Proof. For Z solution to (8.2), recall that $X_t = 2\sqrt{Z_t/\gamma}$ satisfies the SDE $dX_t = dB_t - q(X_t)dt$ with

$$q(x) = \frac{1}{2x} - \frac{2h(\gamma x^2/4)}{\gamma x} \qquad x > 0.$$

In particular we have $q'(x) = -\frac{1}{2x^2} + \frac{2h(\gamma x^2/4)}{\gamma x^2} - h'(\gamma x^2/4)$ and

$$q^2(x) - q'(x) = \frac{3}{4x^2} + h(\gamma x^2/4)\left(\frac{4}{\gamma^2 x^2}h(\gamma x^2/4) - \frac{4}{\gamma x^2}\right) + h'(\gamma x^2/4).$$

Under assumption (HH) we have the following behaviors at 0 and ∞: $q(x) \underset{x\downarrow 0}{\sim} 1/2x$, as well as

$$q^2(x) - q'(x) \underset{x\downarrow 0}{\sim} \frac{3}{4x^2} \qquad \text{and} \qquad (q^2 - q')(2\sqrt{x/\gamma}) \underset{x\to\infty}{\sim} \frac{h(x)^2}{x}\left(\frac{1}{\gamma} + \frac{xh'(x)}{h(x)^2}\right).$$

Then, it is direct to check that hypothesis (H2) holds

$$\lim_{x\to\infty} q^2(x) - q'(x) = \infty \ , \ C := -\inf_{x\in(0,\infty)} q^2(x) - q'(x) < \infty.$$

We recall that

$$Q(x) = \int_1^x 2q(y)dy, \ \Lambda(x) = \int_1^x e^{Q(y)}dy \text{ and } \kappa(x) = \int_1^x e^{Q(y)}\left(\int_1^y e^{-Q(z)}dz\right)dy.$$

Straightforward calculations show that

$$\lim_{x\to\infty} \frac{Q(x)}{x} = \infty \qquad \text{and} \qquad A := \lim_{x\to 0^+} \big(Q(x) - \log(x)\big) \in (-\infty, \infty).$$

In particular, $\Lambda(\infty) = \infty$ and the integrand in the definition of κ is equivalent to $y\log(y)$ which ensures $\kappa(0^+) < \infty$. Thus X, and consequently Z, is absorbed at 0 with probability 1, that is hypothesis (H1) holds.

We now continue with (H3) which is

$$\int_0^1 \frac{1}{q^2(y) - q'(y) + C + 2}\, e^{-Q(y)}dy < \infty.$$

This hypothesis holds because near 0 the integrand is of the order

$$\frac{1}{\frac{3}{4y^2}}e^{-Q(y)} \sim \frac{4e^{-A}}{3}y.$$

For the first part of the Theorem it remains only to show that (H4) holds

$$\int_1^\infty e^{-Q(x)}dx < \infty \quad \text{and} \quad \int_0^1 x\, e^{-Q(x)/2}dx < \infty.$$

The first integral is finite because Q grows at least linearly near ∞ and the second one is finite because the integrand is of order $1/\sqrt{x}$ for x near 0.

Hence we can apply Theorem 5.2, Proposition 5.5, and Corollaries 6.1 and 6.2 to finish with the proof of the first part of the Theorem.

For the last part of the Theorem we need to show that X comes down from infinity which is equivalent to (H5). Thanks to Remark 7.4, there is a simple sufficient condition for this hypothesis to hold, which has three components. The first one

$$q(x) \geq q_0 > 0 \ \text{ for all } x \geq x_0$$

follows from (HH)(i). The second one

$$\limsup_{x\to\infty} q'(x)/2q^2(x) < 1$$

is equivalent to

$$\limsup_{x\to\infty} -\frac{xh'(x)}{h(x)^2} < \frac{2}{\gamma},$$

which clearly follows from (HH)(ii). Finally the third one

$$\int_{x_0}^{\infty} \frac{1}{q(x)}\, dx < \infty,$$

thanks to (HH)(i), is equivalent to

$$\int^{\infty} \frac{-\gamma x}{2h(\gamma x^2/4)}\, dx = \int^{\infty} \frac{1}{-h(z)}\, dz < \infty.$$

This is exactly the extra assumption made in the Theorem and the result is proven. □

8.2. **The growth function and conditioning.** Referring to the previous construction of the generalized Feller diffusion (8.2), we saw why $h(z)$ could be viewed as the expected growth rate of a population of size z and $h(z)/z$ as the mean *per capita* growth rate. Indeed, $h(z)$ informs of the resulting action of density upon the growth of the population, and $h(z)/z$ indicates the resulting action of density upon each individual. In the range of densities z where $h(z)/z$ increases with z, the most important interactions are of the *cooperative* type, one speaks of *positive density-dependence*. On the contrary, when $h(z)/z$ decreases with z, the interactions are of the *competitive* type, and density-dependence is said to be *negative*. In many cases, such as the logistic one, the limitation of resources forces harsh competition in large populations, so that, as $z \to \infty$, $h(z)/z$ is negative and decreasing. In particular $h(z)$ goes to $-\infty$. The shape of h at infinity determines the long time behavior of the diffusion Z.

Actually, if h goes to infinity at infinity, such as in the pure branching process case (where h is linear), Theorem 8.2 still holds if (HH)(i) is replaced with the more general condition $\lim_{x\to\infty} \frac{h(x)}{\sqrt{x}} = \pm\infty$, *provided* the generalized Feller diffusion is further *conditioned on eventual extinction*. Indeed, the following statement ensures that conditioning on extinction roughly amounts to *replacing h with $-h$*.

Proposition 8.3. *assume that Z is given by (8.2), where $h \in C^1([0,\infty))$, $h(0) = 0$, $\lim_{x\to\infty} \frac{h(x)}{\sqrt{x}} = \infty$. Define $u(x) := \mathbb{P}_x(\lim_{t\to\infty} Z_t = 0)$ and let Y be the diffusion Z conditioned on eventual extinction. Then Y is the solution of the SDE, $Y_0 = Z_0$*

$$dY_t = \sqrt{\gamma Y_t} dB_t + \left(h(Y_t) + \gamma Y_t \frac{u'(Y_t)}{u(Y_t)} \right) dt. \tag{8.3}$$

If, in addition h satisfies (HH)(ii) then

$$h(y) + \gamma y \frac{u'(y)}{u(y)} \sim_{y\to\infty} -h(y).$$

Proof. Let $J(x) := \int_0^x \frac{2h(z)}{\gamma z} dz$ which is well-defined since $h \in C^1([0,\infty))$ with $h(0) = 0$. We set

$$v(x) := a \int_x^{\infty} e^{-J(z)} dz,$$

with $a = (\int_0^\infty e^{-J(z)}dz)^{-1}$ (well-defined by the growth of h near ∞). Now we prove that $u = v$. It is easily checked that v is decreasing with $v(0) = 1$, $v(\infty) = 0$, and that it satisfies the equation $\frac{\gamma}{2}xv''(x) + h(x)v'(x) = 0$ for all $x \geq 0$.
As a consequence, $(v(Z_t); t \geq 0)$ is a (bounded hence) uniformly integrable martingale, so that

$$v(x) = \mathbb{E}_x(v(Z_t)) \to v(0)\mathbb{P}_x(\lim_{t\to\infty} Z_t = 0) + v(\infty)\mathbb{P}_x(\lim_{t\to\infty} Z_t = \infty) = u(x),$$

as $t \to \infty$, so that indeed $u = v$.
Using the strong Markov Property of Z we obtain that for any Borel set $A \subset (0, \infty)$ and $s \geq 0$

$$\mathbb{P}_x(Y_s \in A) = \mathbb{P}_x(Z_s \in A \mid T_0 < \infty) = \mathbb{E}_x\left(\frac{\mathbb{P}_{Z_s}(T_0 < \infty)}{\mathbb{P}_x(T_0 < \infty)},\, Z_s \in A\right) = \mathbb{E}_x\left(\frac{u(Z_s)}{u(x)},\, Z_s \in A\right).$$

Then for any measurable and bounded function f we get

$$\mathbb{E}_x(f(Y_s)) = \mathbb{E}_x\left(f(Z_s)\frac{u(Z_s)}{u(x)}\right).$$

Now if f is C^2 and has compact support contained in $(0, \infty)$, we get from Itô's formula that uf is in the domain of L^Z, the generator of Z, and then f is in the domain of the generator L^Y of Y and moreover

$$L^Y(f)(x) = \frac{1}{u(x)}L^Z(uf)(x) = \frac{\gamma}{2}xf''(x) + \left(h(x) + \gamma x\frac{u'(x)}{u(x)}\right)$$

Then, since h is locally Lipschitz we obtain that the law of Y is the unique solution to the SDE (8.3).

Let us show the last part of the proposition. Notice that J is strictly increasing after some x_0, so we consider its inverse φ on $[J(x_0), \infty)$. Next observe that for $x > x_0$,

$$-\frac{u}{u'}(x) = e^{J(x)}\int_x^\infty e^{-J(z)}dz = e^{J(x)}\int_{J(x)}^\infty e^{-b}\varphi'(b)db,$$

with the change $b = J(z)$. As a consequence, we can write for $y > J(x_0)$

$$-\frac{u}{u'}(\varphi(y)) = e^y\int_y^\infty e^{-b}\varphi'(b)db = \int_0^\infty e^{-b}\varphi'(y+b)db. \tag{8.4}$$

Because h tends to ∞, $J(x) \geq (1+\varepsilon)\log(x)$ for x sufficiently large, so that $\varphi(y) \leq \exp(y/(1+\varepsilon))$, and $\varphi(y)\exp(-y)$ vanishes as $y \to \infty$. Now, since

$$\varphi'(y) = \frac{\gamma\varphi(y)}{2h(\varphi(y))} = o(\varphi(y)),$$

$\varphi'(y)\exp(-y)$ also vanishes. Since h is differentiable, J is twice differentiable, and so is φ, so performing an integration by parts yields

$$\varphi'(y) = \int_0^\infty e^{-b}\varphi'(y+b)db - \int_0^\infty e^{-b}\varphi''(y+b)db. \tag{8.5}$$

Since $\varphi'(J(x)) = 1/J'(x)$, we get $J'(x)\varphi''(J(x)) = (1/J'(x))'$, so by the technical assumption (HH)(ii),

$$\varphi''(J(x)) = \varphi'(J(x))\left(\frac{1}{J'(x)}\right)' = \frac{\gamma}{2}\varphi'(J(x))\left(\frac{1}{h(x)} - \frac{xh'(x)}{h(x)^2}\right) = o\left(\varphi'(J(x))\right),$$

as $x \to \infty$. Then, as $y \to \infty$ we have $\varphi''(y) = o(\varphi'(y))$. This shows, thanks to (8.5), that

$$\int_0^\infty e^{-b}\varphi'(y+b)db \sim_{y\to\infty} \varphi'(y)$$

which entails, thanks to (8.4), that

$$-\frac{u}{u'}(\varphi(y)) \sim_{y\to\infty} \varphi'(y).$$

This is equivalent to

$$\gamma x \frac{u'}{u}(x) \sim_{x\to\infty} -\gamma x J'(x) = -2h(x),$$

which ends the proof. □

Let us examine the case of the Feller diffusion (pure branching process), where $h(z) = rz$. First, it is known (see e.g. [23, Chapter 2]) that when $r > 0$, the supercritical Feller diffusion Z conditioned on extinction is *exactly* the subcritical Feller diffusion with $h(z) = -rz$. The previous statement can thus be seen as an extension of this duality to more general population diffusion processes.
Second, in the critical case ($r = 0$), our present results do not apply. Actually, the (critical) Feller diffusion has no *q.s.d.* [22]. Third, in the subcritical case ($r < 0$), our results do apply, so there is a Yaglom limit and a Q-process, but in contrast to the case when $1/h$ is integrable at ∞, it is shown in [22] that subcritical Feller diffusions have infinitely many *q.s.d.*.

Appendix A. **Proof of Lemma 4.5**

We first prove the second bound. For any nonnegative and continuous function f with support in $\mathbb{R}^+$ we have from hypothesis (H2)

$$\int \tilde{p}_1(x,u)\, f(u)du = \mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\, \mathbb{1}_{1<T_0}(\omega)\, \exp\left(-\frac{1}{2}\int_0^1 (q^2-q')(\omega_s)ds\right)\right]$$
$$\leq e^{C/2}\mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\, \mathbb{1}_{1<T_0}(\omega)\right] .$$

The estimate (4.3) follows by letting $f(z)dz$ tend to the Dirac measure at y with $K_3 = e^{C/2}$, that is

$$\tilde{p}_1(x,y) \leq K_3\, p_1^D(x,y).$$

Here $p_1^D(x,y) = \frac{1}{\sqrt{2\pi}}\left(e^{-(x-y)^2/2} - e^{-(x+y)^2/2}\right)$ (see for example [18, page 97]).

Let us now prove the upper bound in (4.2). Let B_1 be the function defined by

$$B_1(z) := \inf_{u\geq z}\left(q^2(u) - q'(u)\right) .$$

We have

$$\int \tilde{p}_1(x,y)\, f(y)dy = \mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\, \mathbb{1}_{1<T_0}\mathbb{1}_{1<T_{x/3}}\, \exp\left(-\frac{1}{2}\int_0^1 (q^2-q')(\omega_s)ds\right)\right]$$
$$+\mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\, \mathbb{1}_{1<T_0}\, \mathbb{1}_{1\geq T_{x/3}} \exp\left(-\frac{1}{2}\int_0^1 (q^2-q')(\omega_s)ds\right)\right] .$$

For the first expectation we have

$$\mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\,\mathbb{1}_{1<T_0}\mathbb{1}_{1<T_{x/3}}\exp\left(-\frac{1}{2}\int_0^1(q^2-q')(\omega_s)ds\right)\right]$$
$$\le e^{-B_1(x/3)/2}\,\mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\,\mathbb{1}_{1<T_0}\right]\ .$$

For the second expectation, we obtain

$$\mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\,\mathbb{1}_{1<T_0}\,\mathbb{1}_{1\ge T_{x/3}}\exp\left(-\frac{1}{2}\int_0^1(q^2-q')(\omega_s)ds\right)\right]$$
$$\le e^{C/2}\,\mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\,\mathbb{1}_{1<T_0}\mathbb{1}_{1\ge T_{x/3}}\right]$$
$$=e^{C/2}\left(\mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\,\mathbb{1}_{1<T_0}\right]-\ \mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\,\mathbb{1}_{1<T_{x/3}}\right]\right)\ .$$

Using a limiting argument as above and the invariance by translation of the law of the Brownian motion, and firstly assuming that $y/2<x<2y$, we obtain

$$\tilde{p}_1(x,y)\le e^{-B_1(x/3)/2}p_1^D(x,y)+e^{C/2}\left(p_1^D(x,y)-p_1^D(2x/3,y-x/3)\right)\ ,$$

$$p_1^D(x,y)-p_1^D(2x/3,y-x/3)=\frac{1}{\sqrt{2\pi}}\left(e^{-(y+x/3)^2/2}-e^{-(x+y)^2/2}\right)\le\frac{1}{\sqrt{2\pi}}\,e^{-\max\{x,y\}^2/18}\ .$$

Since the function B_1 is non-decreasing, we get for $y/2<x<2y$

$$\tilde{p}_1(x,y)\le\frac{1}{\sqrt{2\pi}}\left(e^{-B_1(\max\{x,y\}/6)/2}+e^{-\max\{x,y\}^2/18}\right)\ .$$

If $x/y\notin]1/2,2[$, we get from the estimate (4.3)

$$\tilde{p}_1(x,y)\le\frac{K_3}{\sqrt{2\pi}}\,e^{-(y-x)^2/2}\le\frac{K_3}{\sqrt{2\pi}}e^{-\max\{x,y\}^2/8}\ .$$

We now define the function B by

$$B(z):=\log\left(\frac{K_3\vee 1}{\sqrt{2\pi}}\right)+\min\left\{B_1(z/6)/4\ ,\ z^2/36\right\}\ .$$

It follows from hypothesis (H2) that $\lim_{z\to\infty}B(z)=\infty$. Combining the previous estimates we get for any x and y in $\mathbb{R}^+$

$$\tilde{p}_1(x,y)\le e^{-2B(\max\{x,y\})}\ .$$

The upper estimate (4.2) follows by taking the geometric average of this result and (4.3). We now prove that $\tilde{p}_1(x,y)>0$. For this purpose, let $a=\min\{x,y\}/2$ and $b=2\max\{x,y\}$. We have as above for every nonnegative continuous function f with support in $\mathbb{R}^+$

$$\int\tilde{p}_1(x,y)\ f(y)dy\ge\mathbb{E}^{\mathbb{W}_x}\left[f(\omega(1))\,\mathbb{1}_{1<T_{[a,b]}}\exp\left(-\frac{1}{2}\int_0^1(q^2-q')(\omega_s)ds\right)\right]$$

where we denote $T_{[a,b]}$ the exit time from the interval $[a,b]$. Let

$$R_{a,b}=\sup_{x\in[a,b]}\left(q^2(x)-q'(x)\right)\ ,$$

this quantity is finite since $q \in C^1((0,\infty))$. We obtain immediately

$$\int \tilde{p}_1(x,y)\, f(y)dy \geq e^{-R_{a,b}/2} \int p_1^{[a,b]}(x,y)\, f(y)dy$$

where we denote $p_t^{[a,b]}$ the heat kernel with Dirichlet conditions in $[a,b]$. The result follows from a limiting argument as above since $p_1^{[a,b]}(x,y) > 0$. □

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